Approximation Algorithms for Polynomial-Expansion and Low-Density Graphs

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Gameplan

1. *Pregame*

   - definition of problems   - a hardness result

2. *Low-density objects and graphs*

   - basic properties   - overview of results

3. *Polynomial expansion*

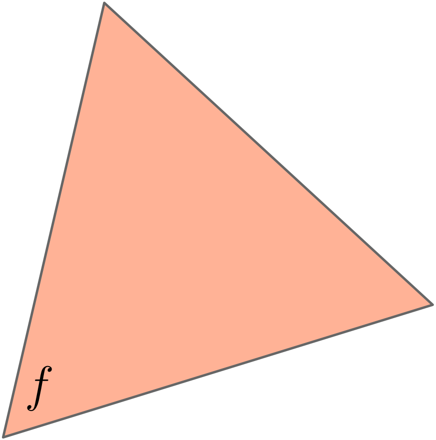
   - basic properties   - overview of results

halftime!

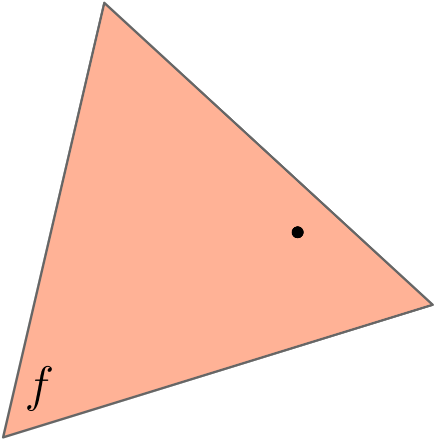
4. *Two proofs*

   - independent set   - dominating set

Fat objects



Fat objects



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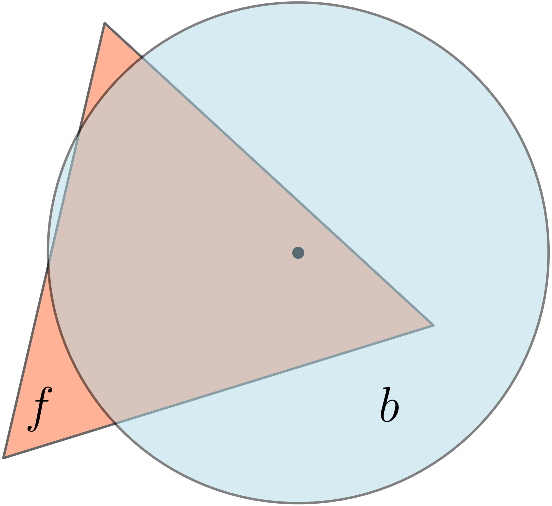

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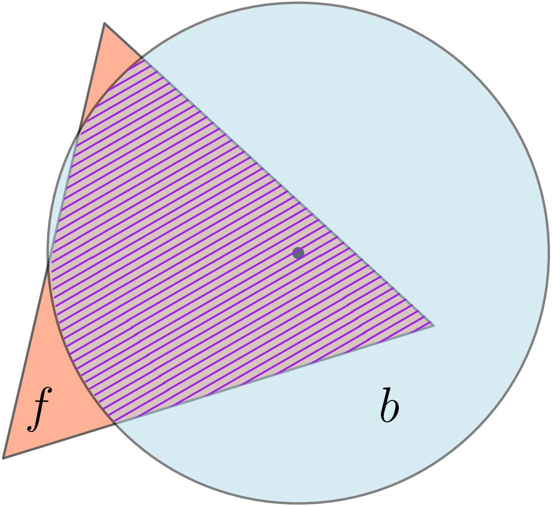

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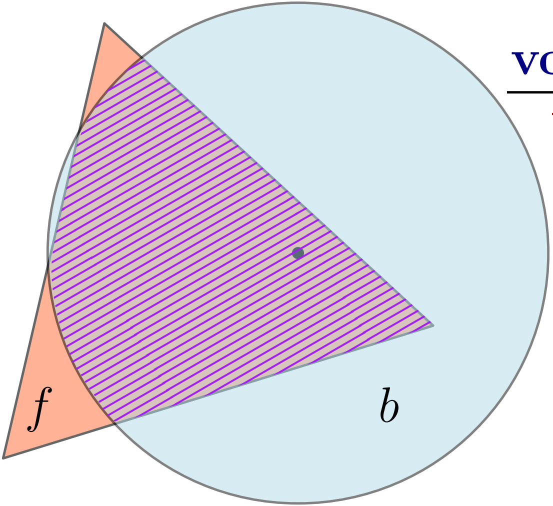
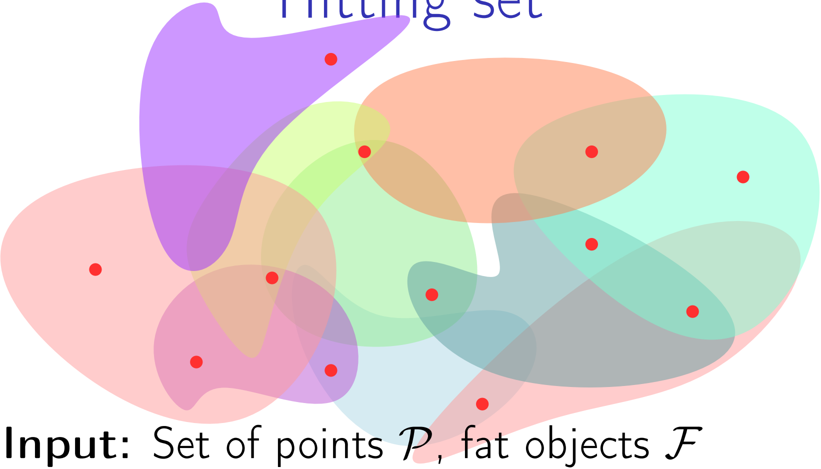


$$\frac{\mathbf{vol}(b \cap f)}{\mathbf{vol}(b)} = \Omega(1)$$

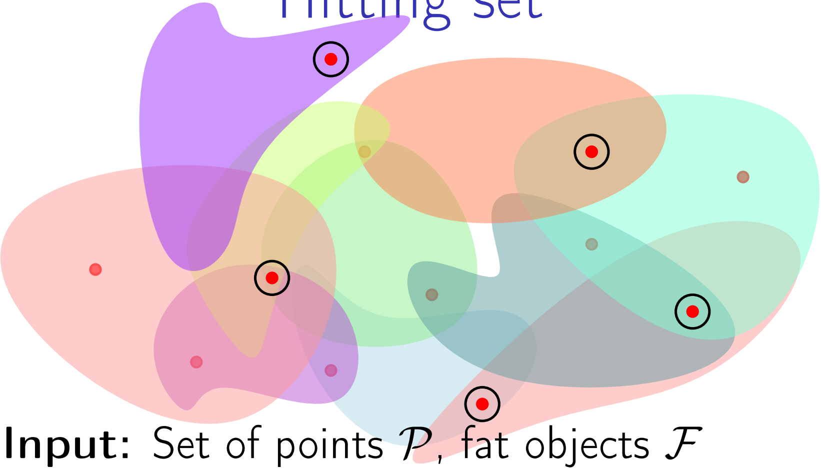
Hitting set



Input: Set of points $\mathcal{P}$, fat objects $\mathcal{F}$

Output: The smallest cardinality subset of $\mathcal{P}$ that pierces every object in $\mathcal{F}$.

Hitting set



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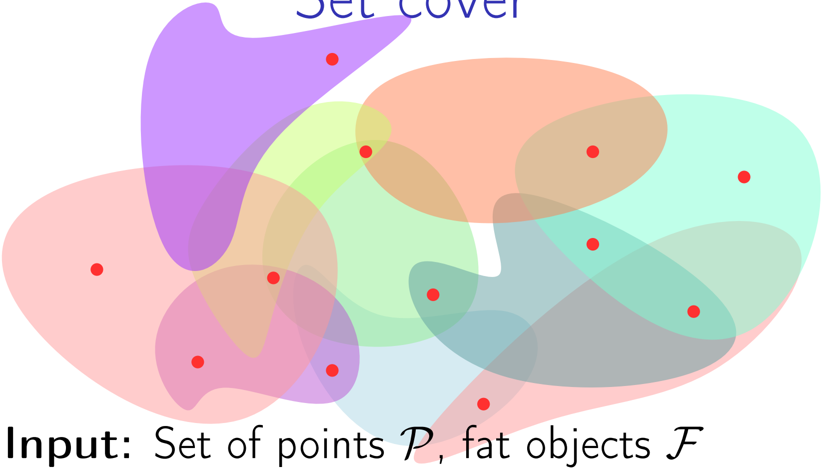
Set cover



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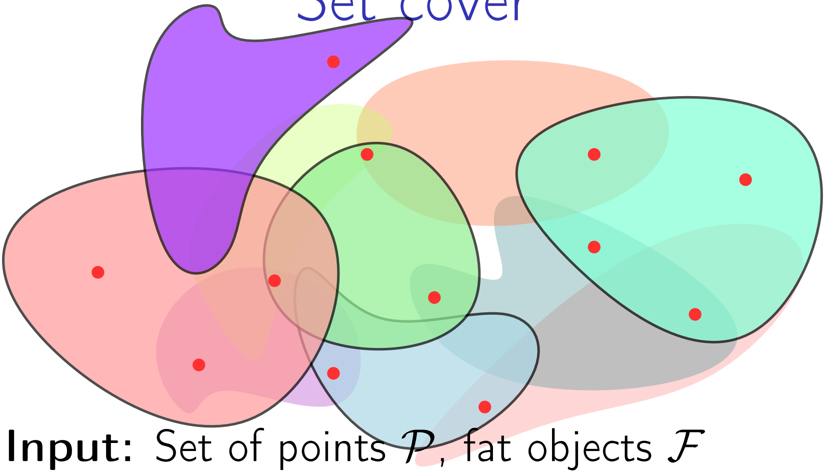

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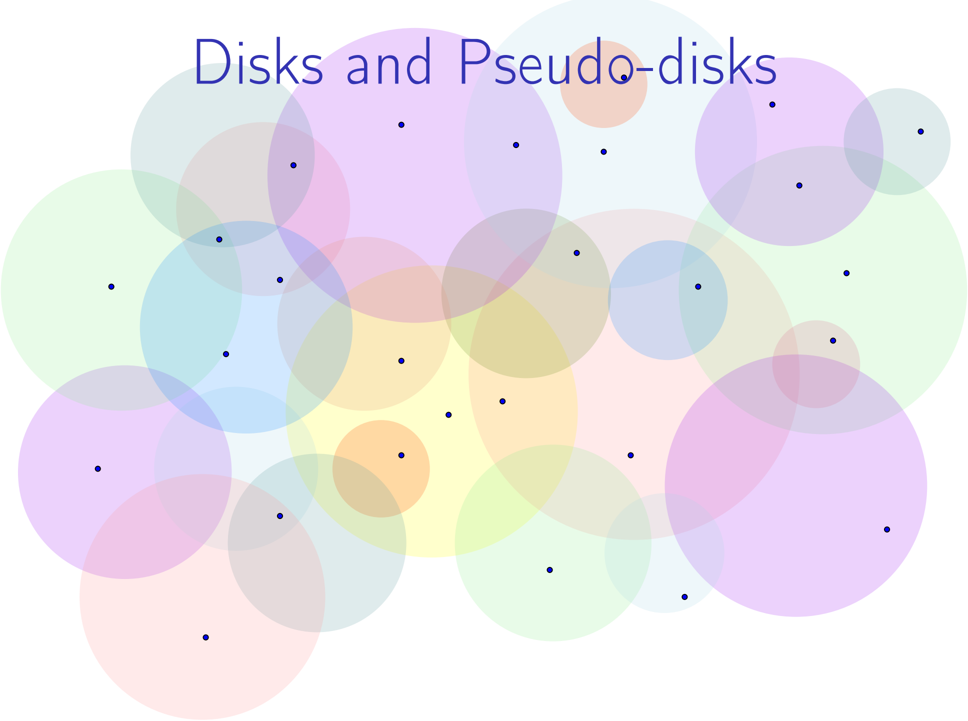
Disks and Pseudo-disks



Disks and Pseudo-disks

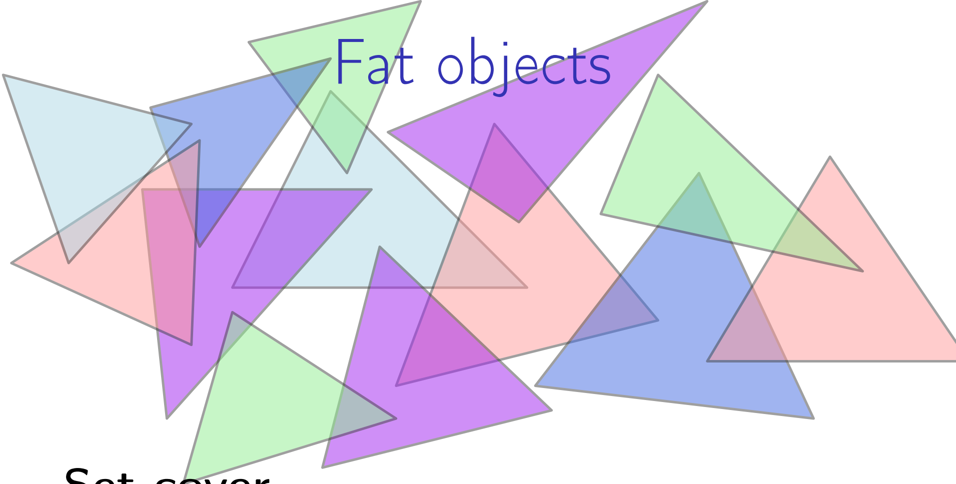
Set Cover

NP-Hard	Feder and Greene 1988
PTAS	Mustafa, Raman and Ray 2014

Hitting Set

NP-Hard	Feder and Greene 1988
PTAS	Mustafa and Ray 2010

Fat objects

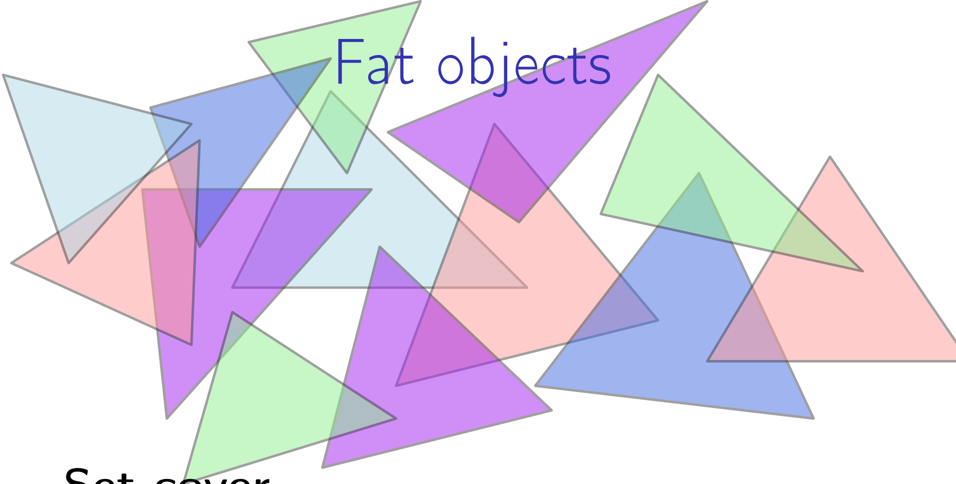


Set cover

$O(1)$ approx. for fat triangles of same size

Clarkson and Varadarajan 2007

Fat objects

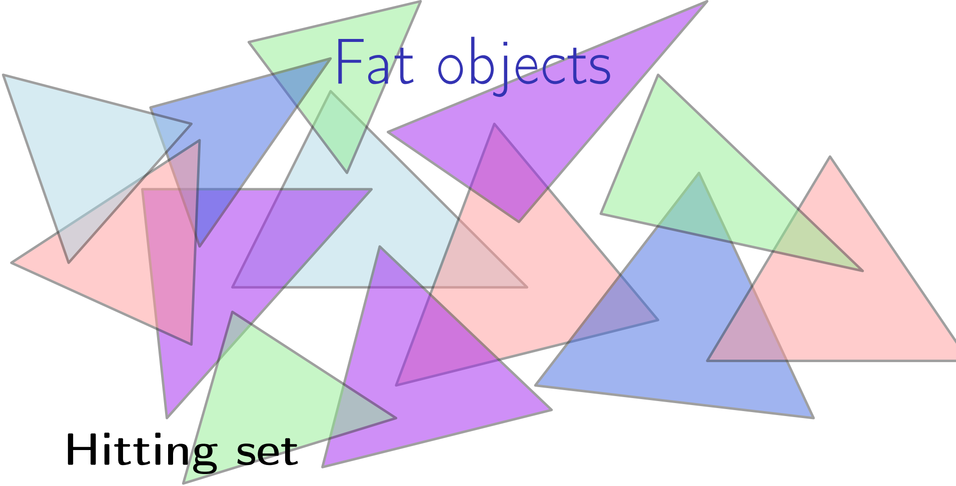


Set cover

$O(\log^* \mathrm{OPT})$ for fat objects in $\mathbb{R}^2$

Aronov, de Berg, Ezra and Sharir 2014

Fat objects



Hitting set

$O(\log\log \mathrm{OPT})$ for fat triangles of similar size

Aronov, Ezra and Sharir 2010

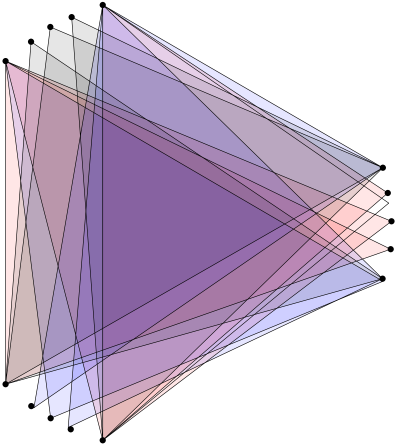
Fat, nearly equilateral triangles

Set cover

- APX-Hard

Hitting set

- APX-Hard



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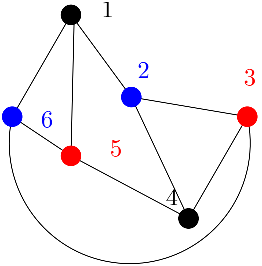

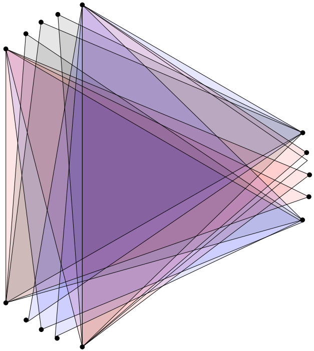

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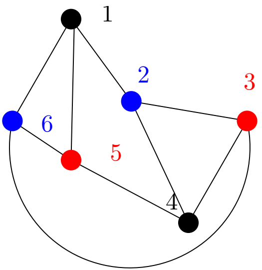

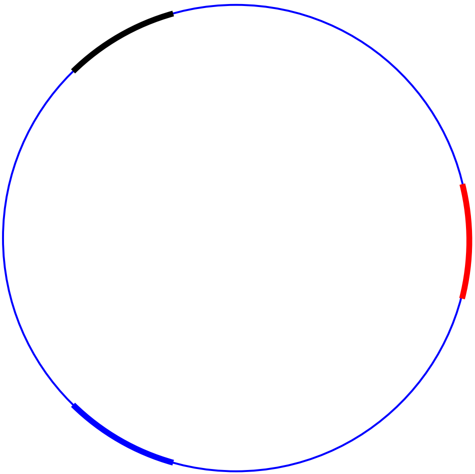

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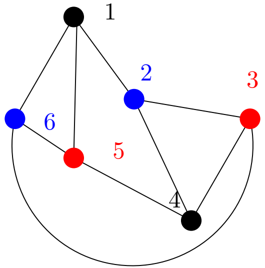

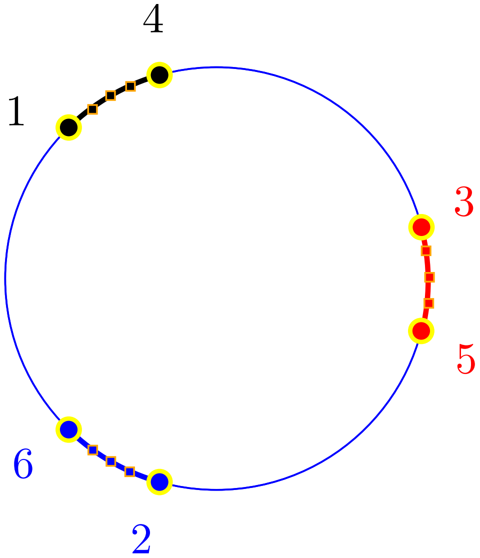

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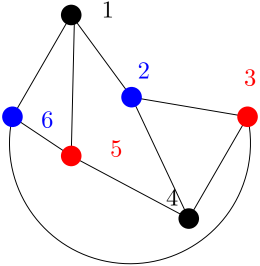

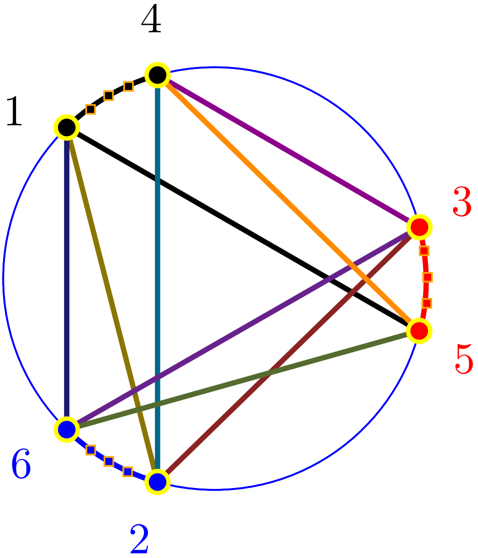

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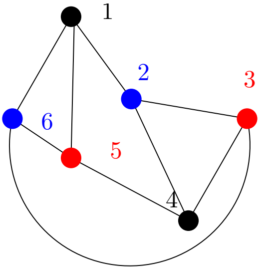

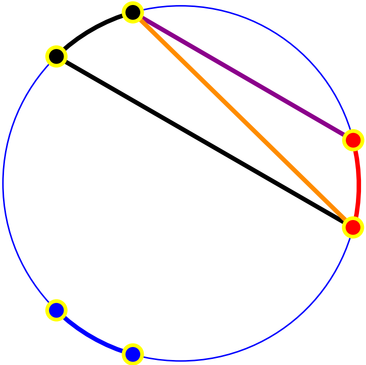

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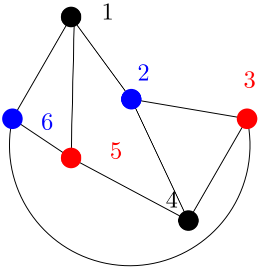

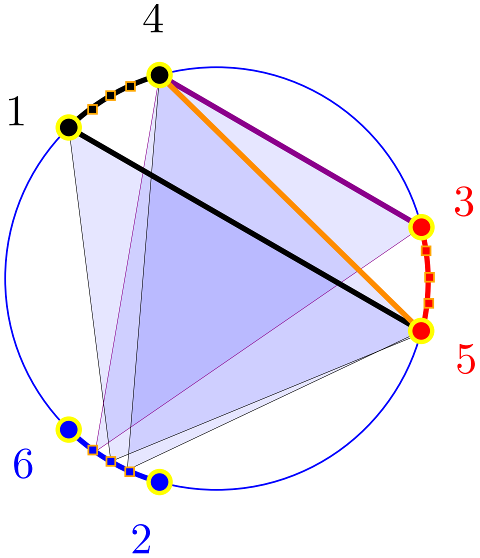

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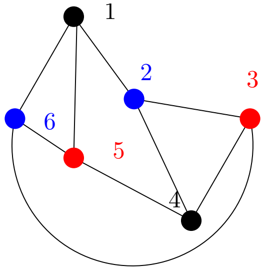

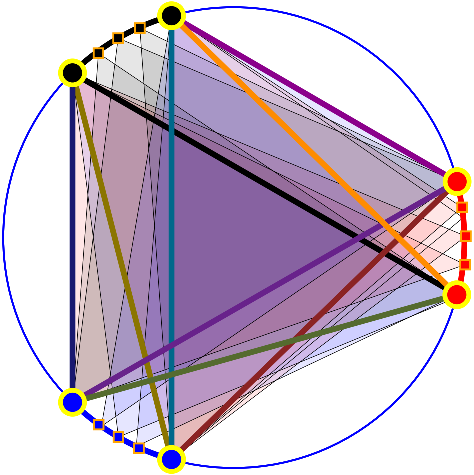

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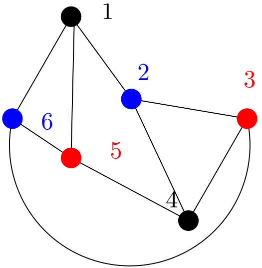
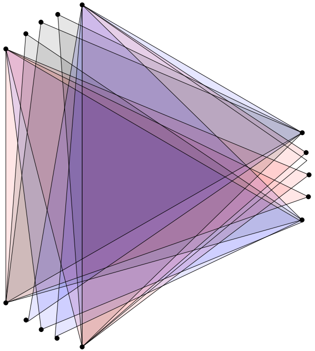
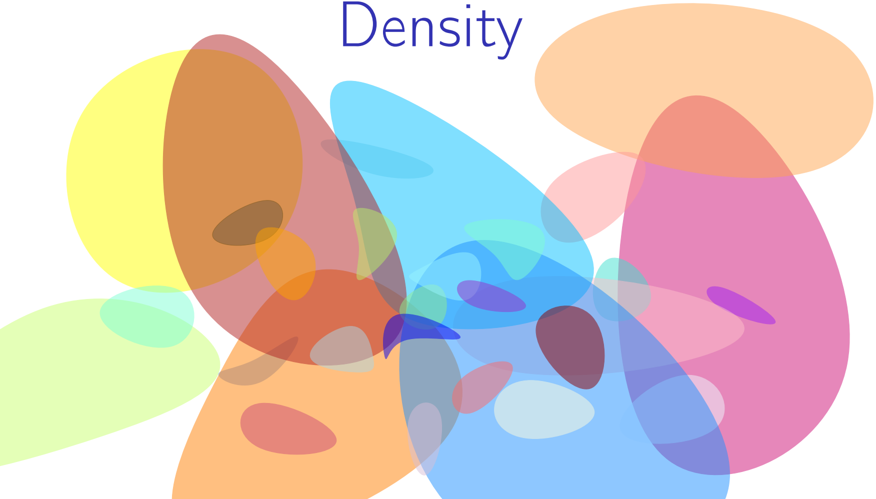




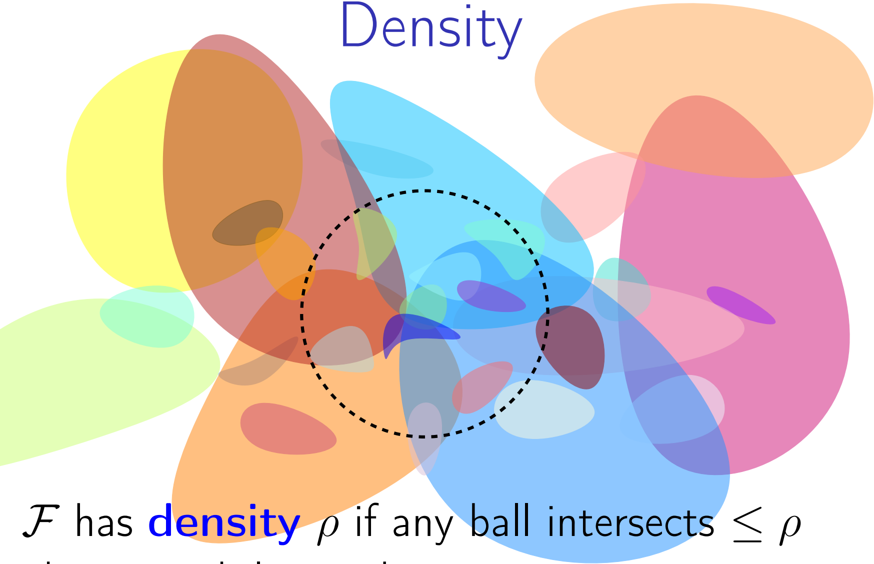
Density



$\mathcal{F}$ has **density** ρ if any ball intersects $\leq \rho$ objects with larger diameter.

van der Stappen, Overmars, de Berg, Vleugels, 1998

Density



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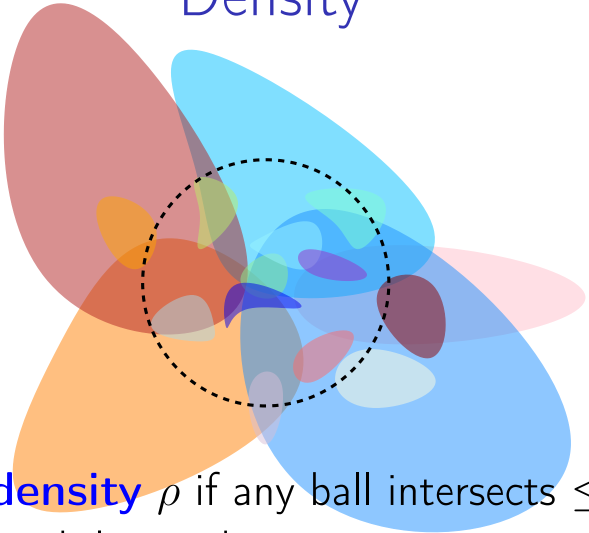

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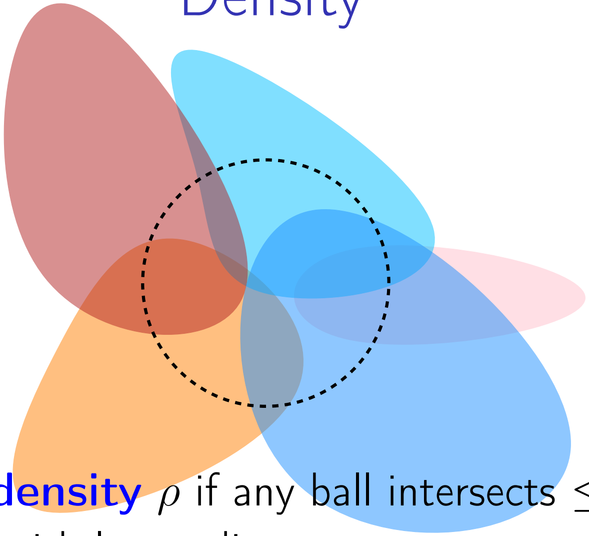

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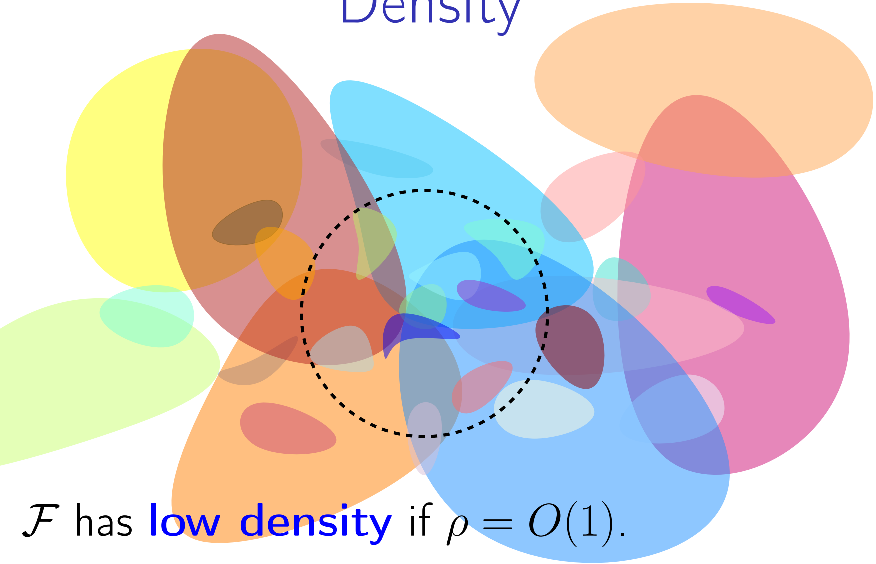
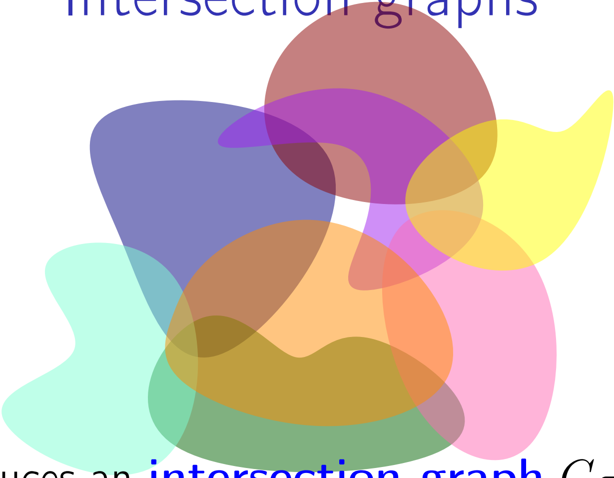


$\mathcal{F}$ has **low density** if $\rho = O(1)$.

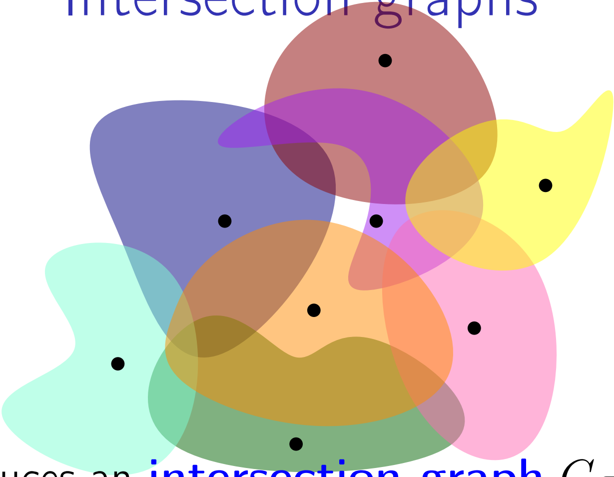
van der Stappen, Overmars, de Berg, Vleugels, 1998

Intersection graphs



$\mathcal{F}$ induces an **intersection graph** $G_{\mathcal{F}}$ with objects as vertices and edges representing overlap.

Intersection graphs



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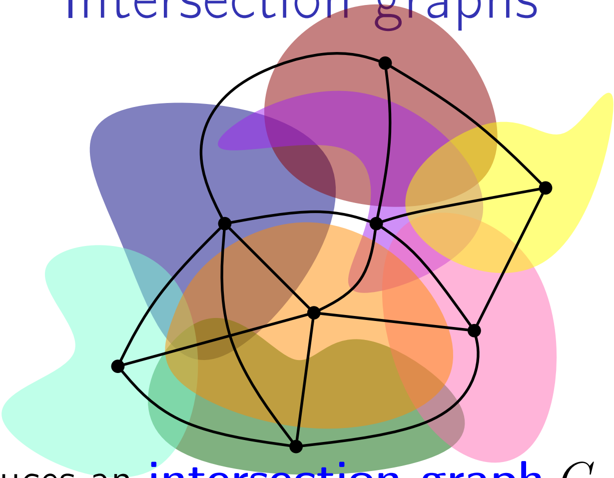

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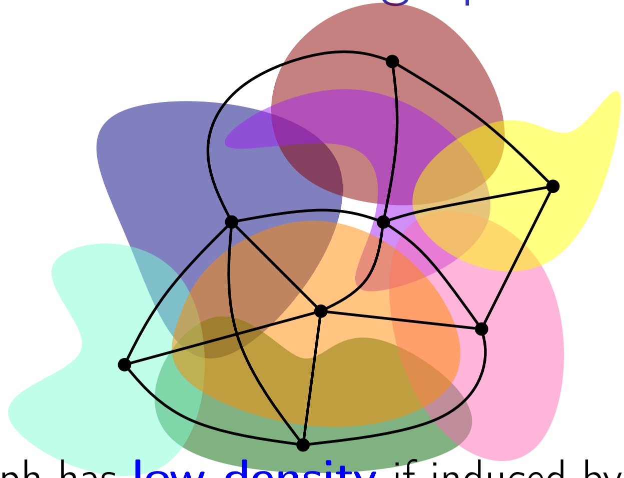
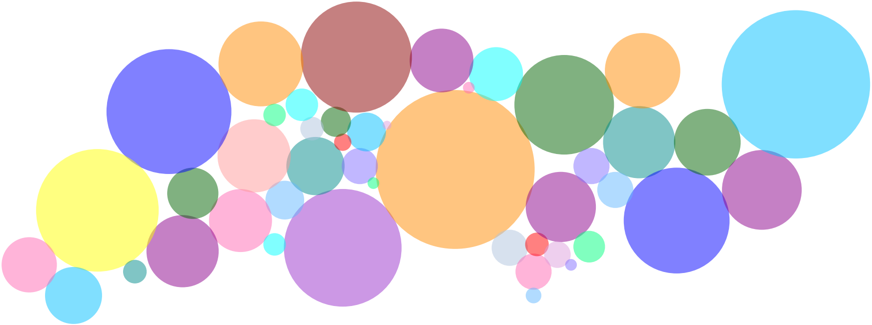


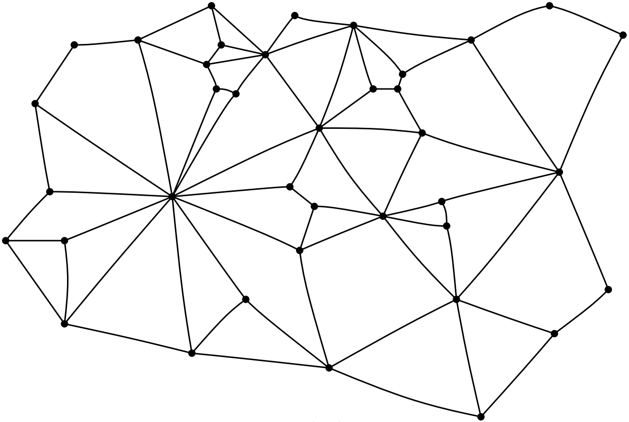
A graph has **low-density** if induced by low-density objects.

Examples of low density



Interior disjoint disks have $O(1)$ density.

Examples of low density



Planar graphs have $O(1)$-density via Circle Packing Theorem.

Koebe, Andreev, Thurston

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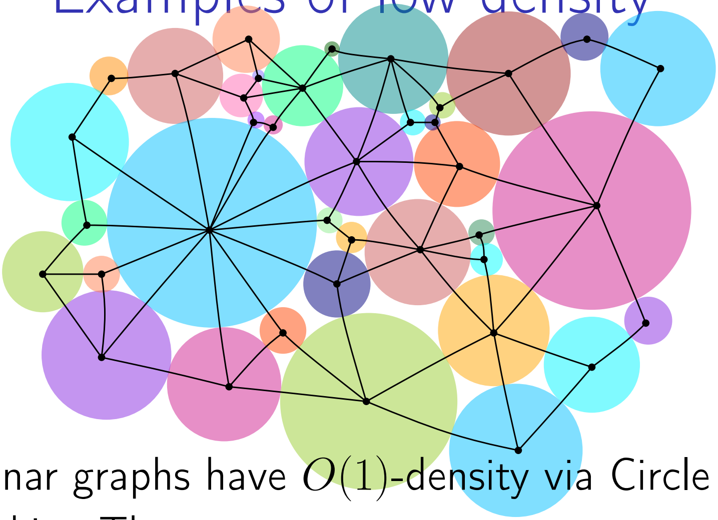

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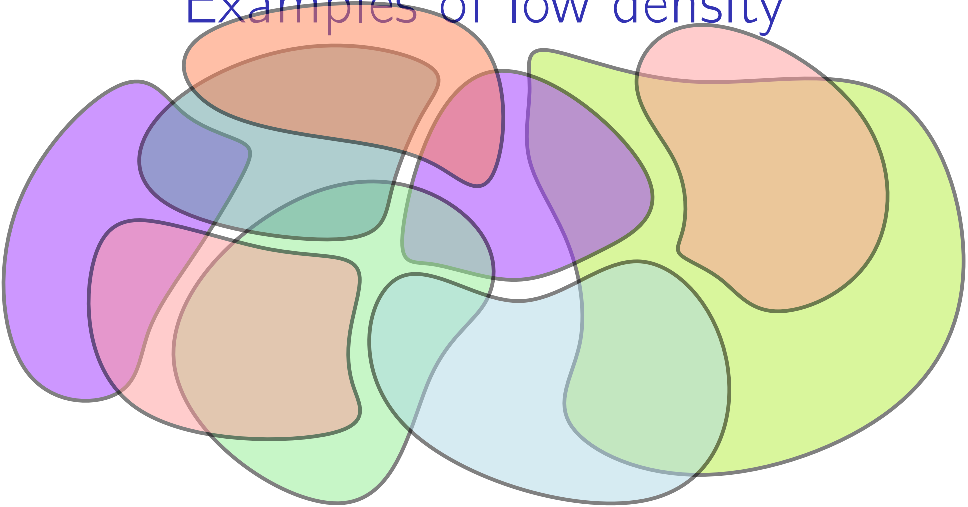
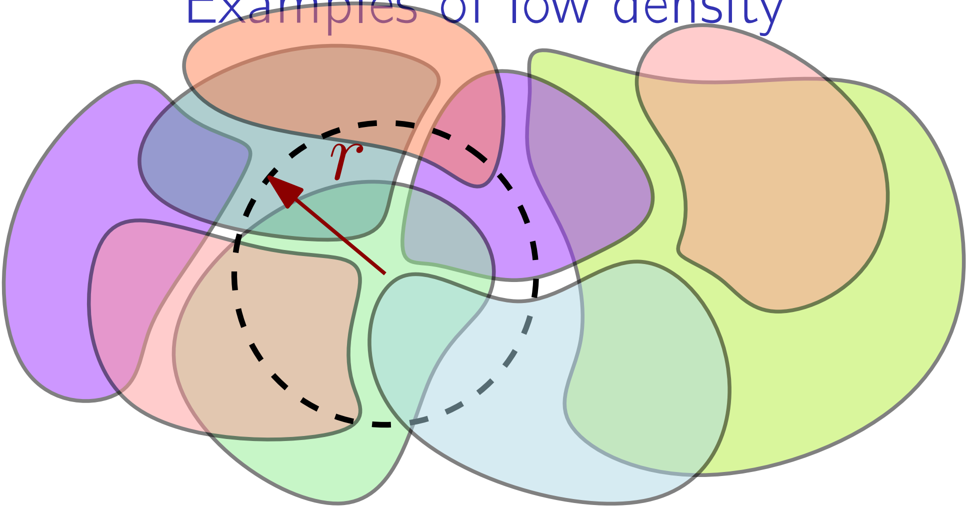


Fat convex objects in $\mathbb{R}^d$ with depth k have density $O(k2^d)$.

Examples of low density



Fat convex objects in $\mathbb{R}^d$ *with depth* k *have density* $O(k2^d)$.

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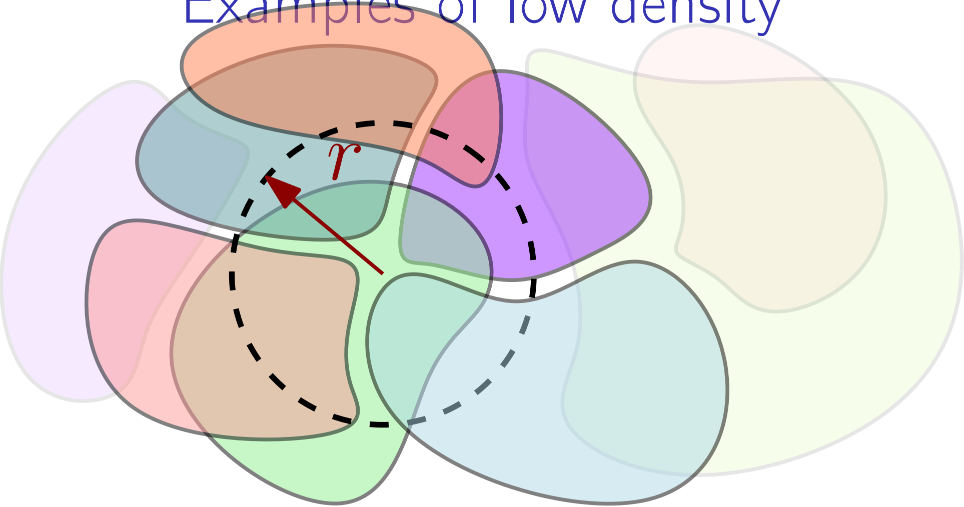

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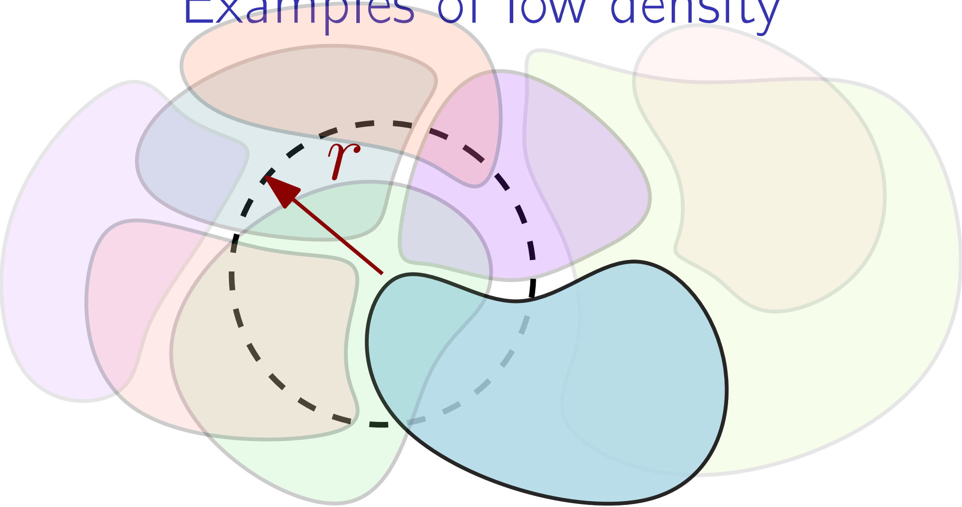

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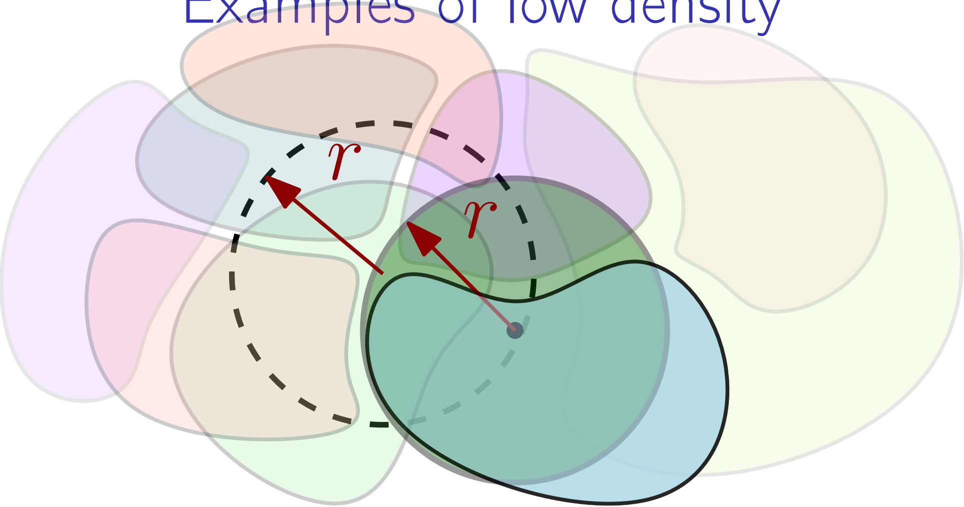

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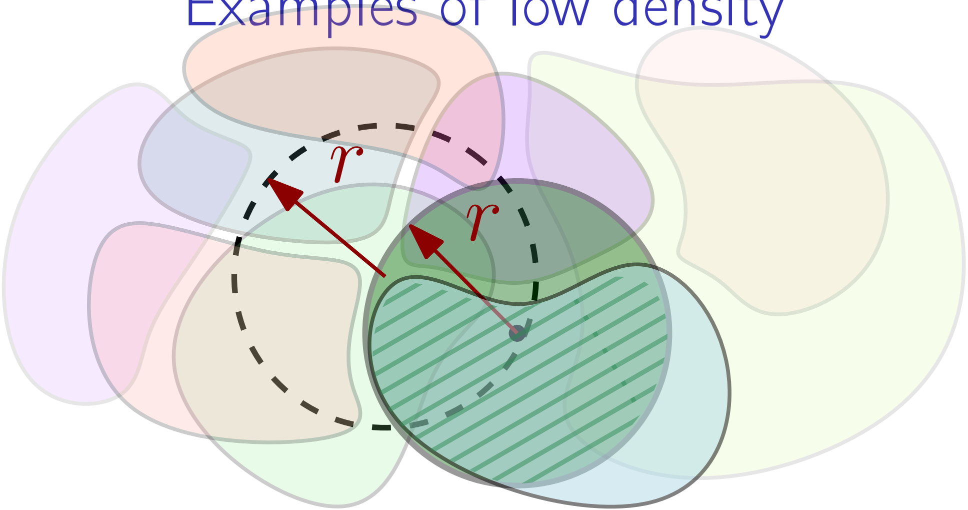

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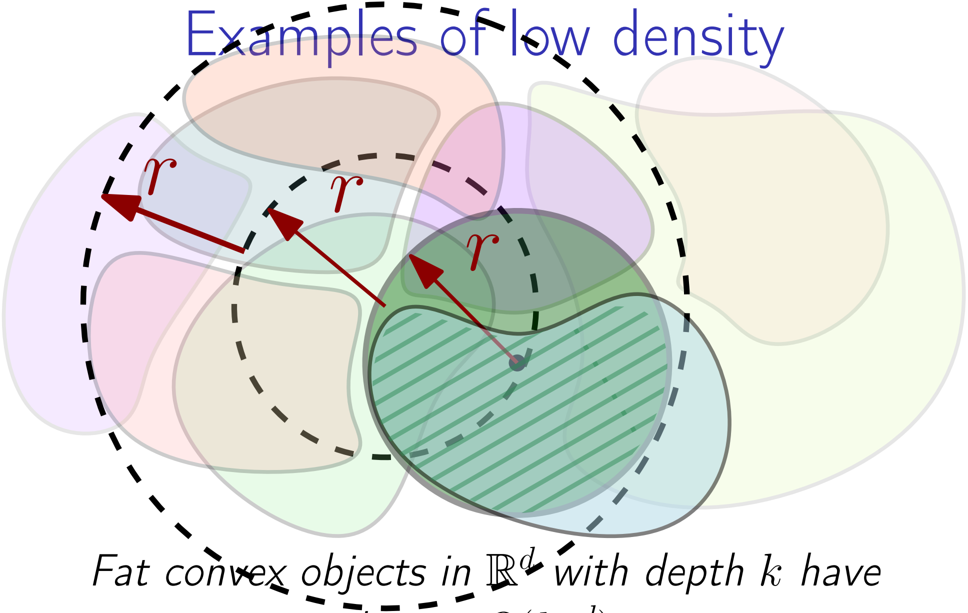

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Additivity

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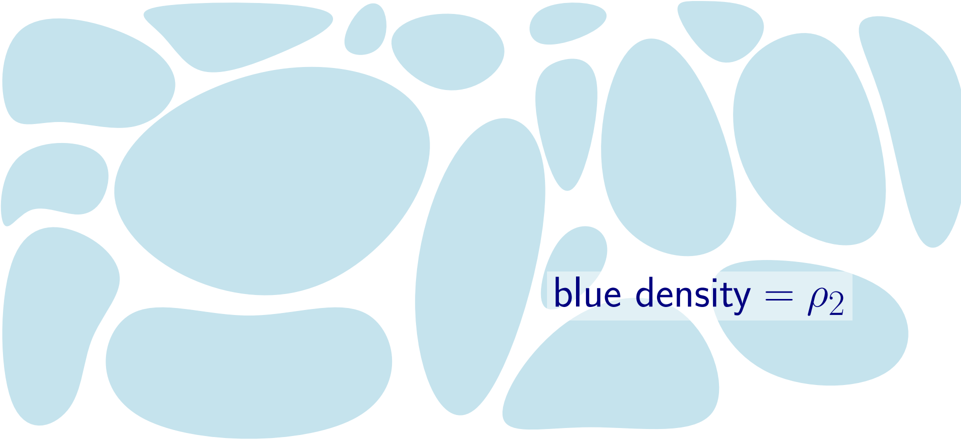

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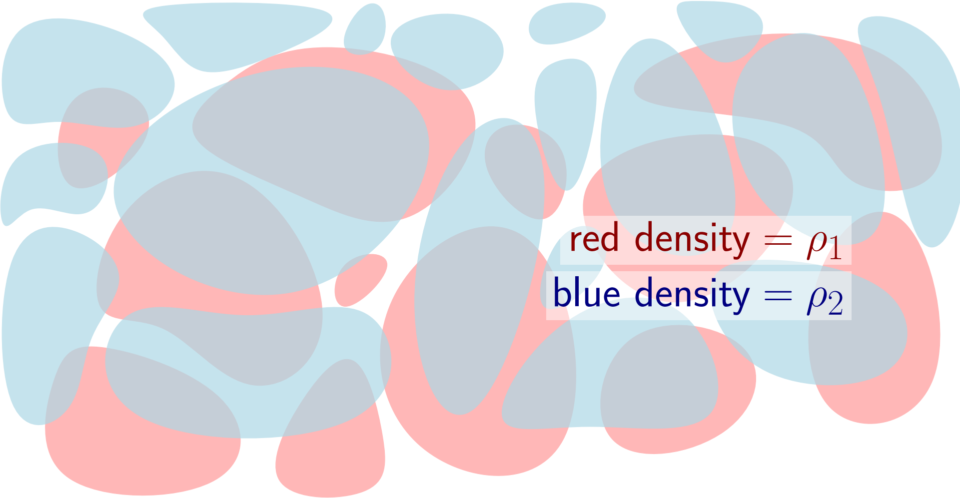

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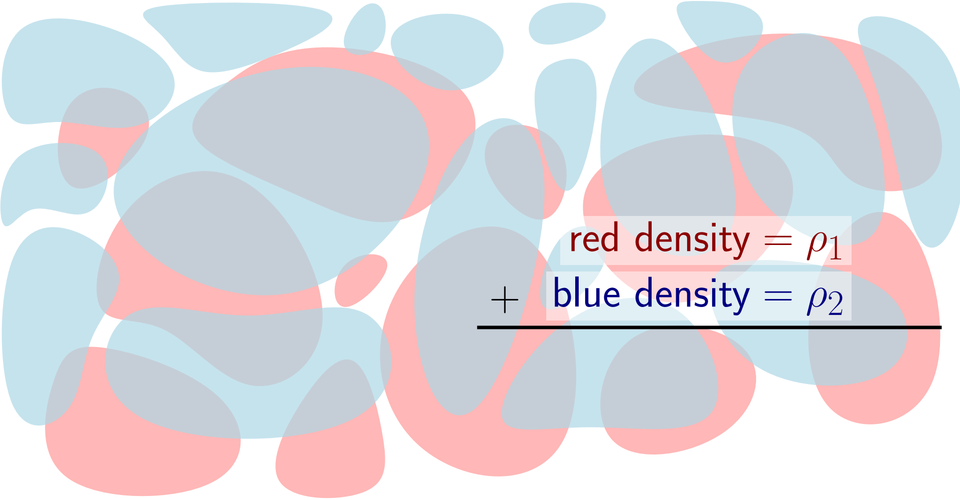

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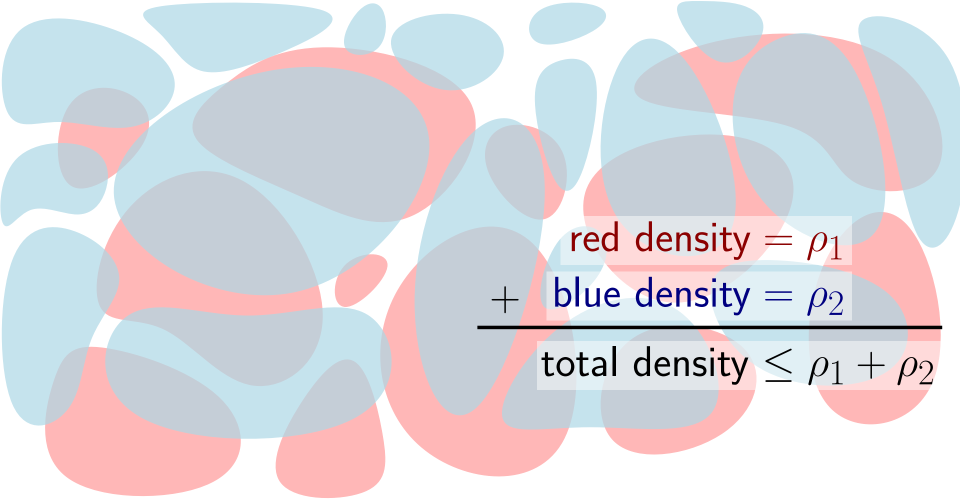
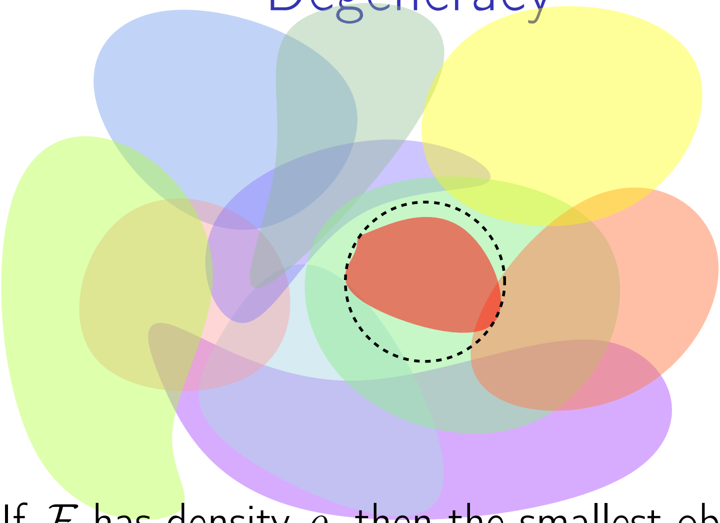


$$
\begin{array}{rl}
 & \text{red density} = \rho_1 \\
+ & \text{blue density} = \rho_2 \\
\hline
 & \text{total density} \leq \rho_1 + \rho_2
\end{array}
$$

Degeneracy

If $\mathcal{F}$ has density ρ, then the smallest object intersects at most $\rho - 1$ other objects.

Degeneracy



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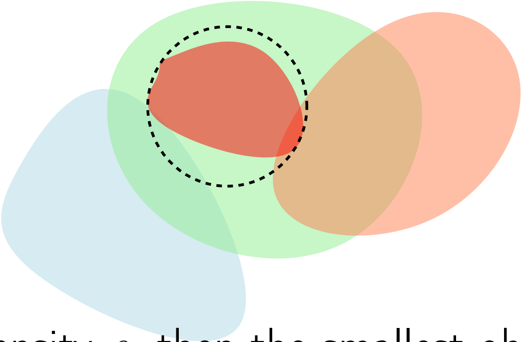

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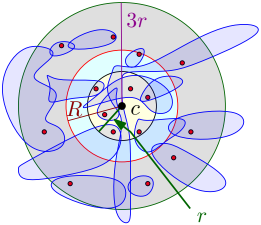
Separators

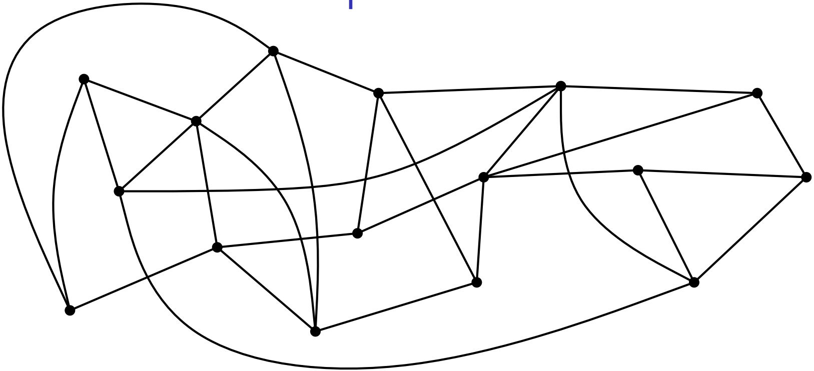
For $k \leq |\mathcal{F}|$, compute a sphere S that

- Strictly contains at least $k - o(k)$ objects and at most k objects.
- Intersects $O(\rho + \rho^{1/d} k^{1-1/d})$ objects.



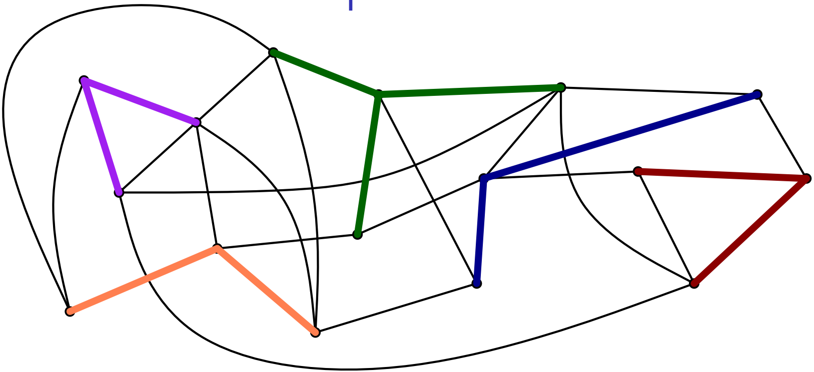
Miller, Teng, Thurston, and Vavasis, 1997; Smith and Wormald, 1998; Chan 2003

Graph minors



A **minor** of G is a graph H obtained by contracting edges, deleting edges, and deleting vertices.

Graph minors



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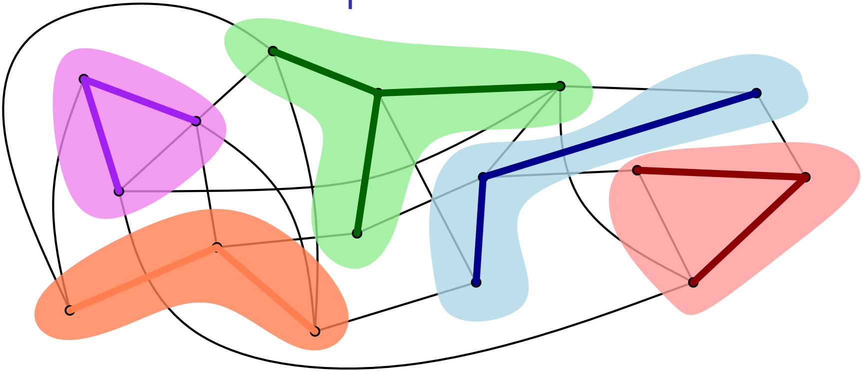

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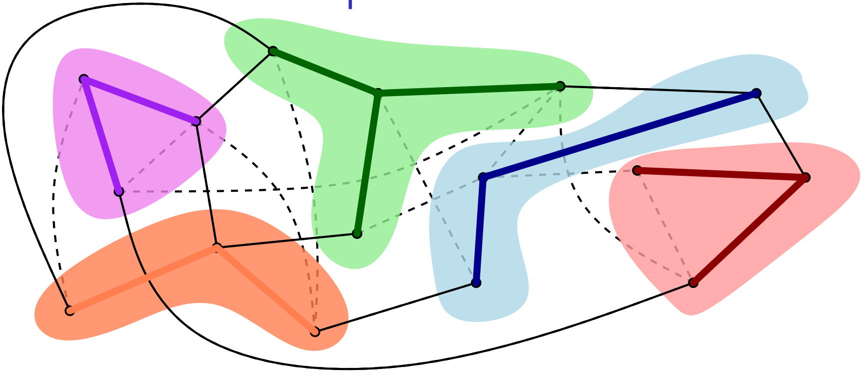
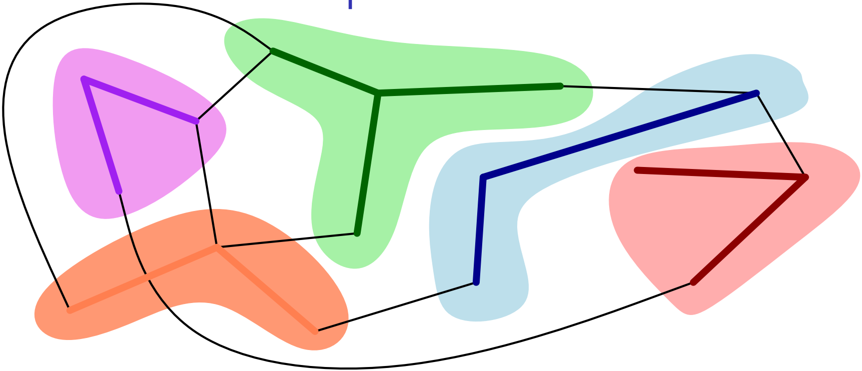
Graph minors



A **minor** of G is a graph H obtained by contracting edges, deleting edges, and deleting vertices.

Graph minors



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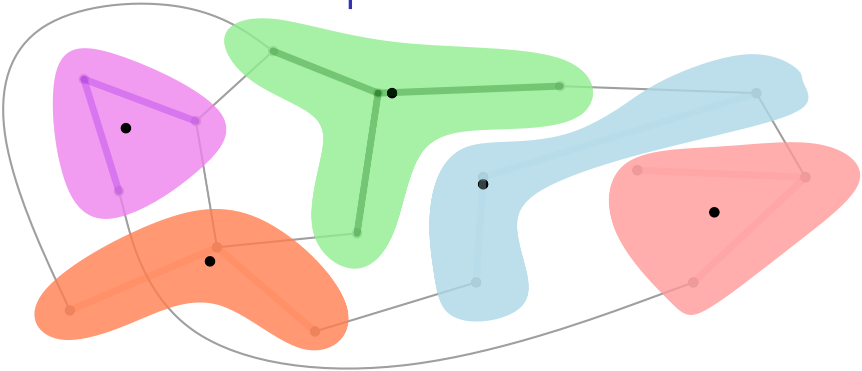
Graph minors



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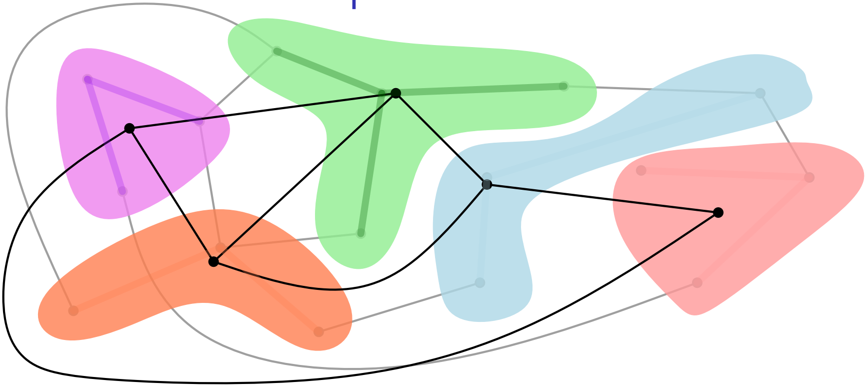
Graph minors



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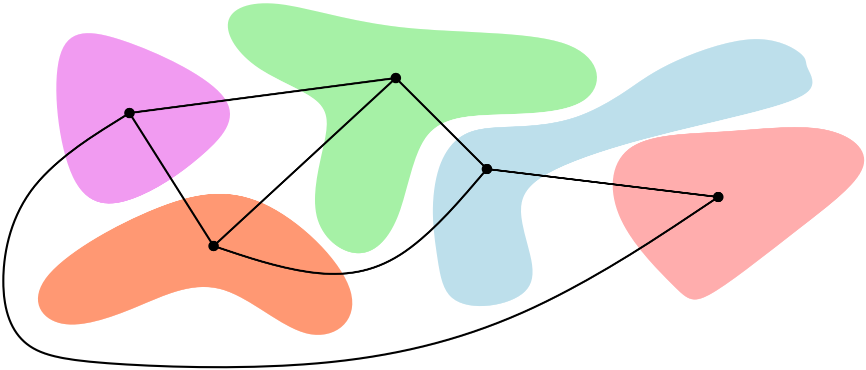
Graph minors



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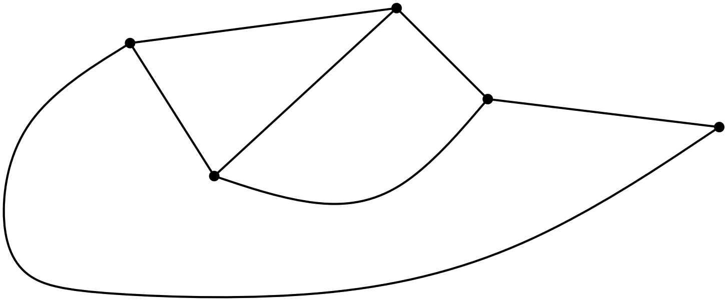
Graph minors



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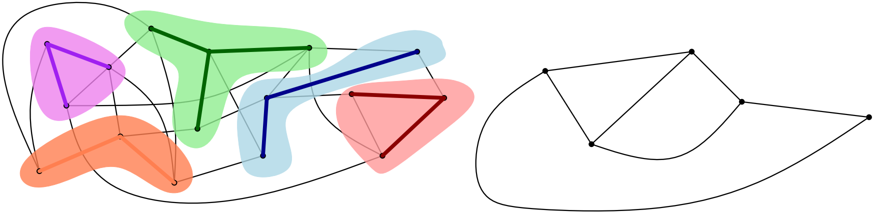
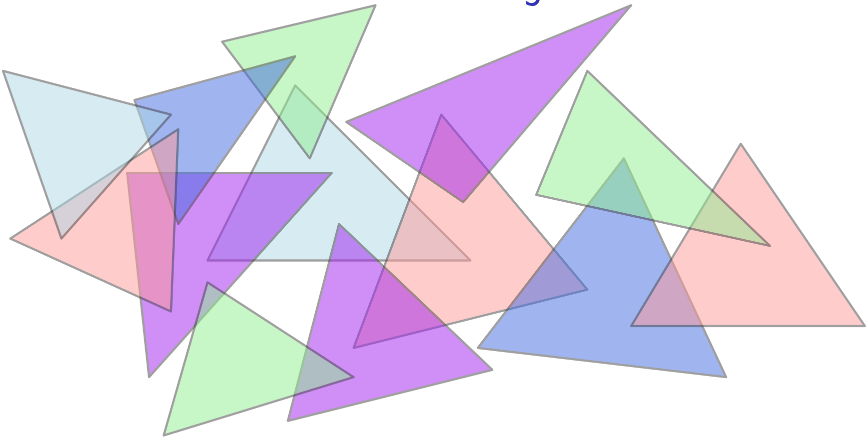
Graph minors



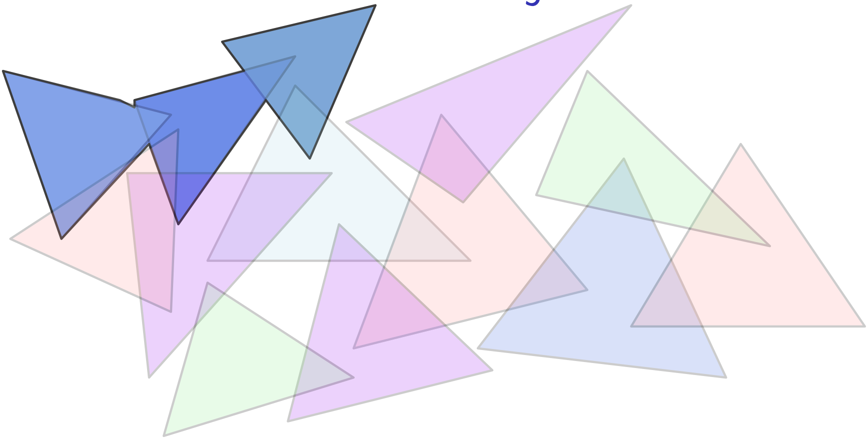
A **minor** of G is a graph H obtained by contracting edges, deleting edges, and deleting vertices.

Minors of objects



$\mathcal{G}$ is a **minor** of $\mathcal{F}$ if it can be obtained by deleting objects and taking unions of overlapping of objects.

Minors of objects



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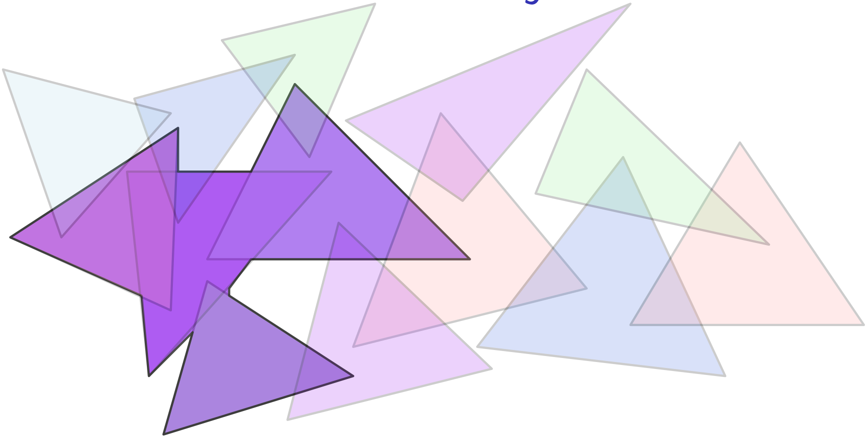

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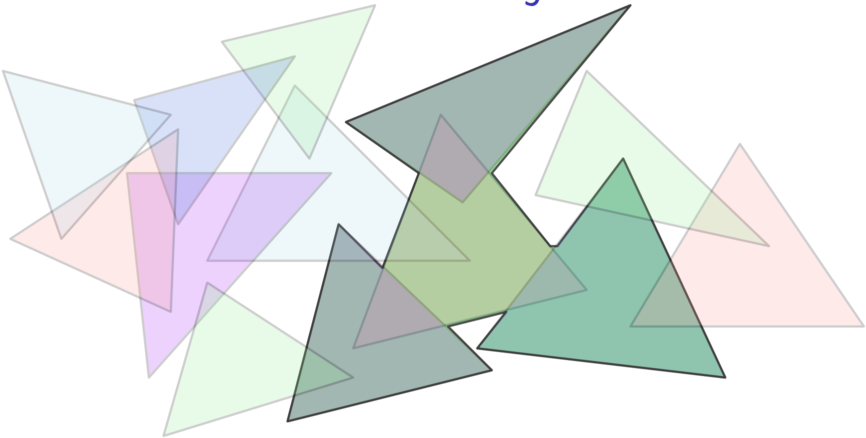
Minors of objects



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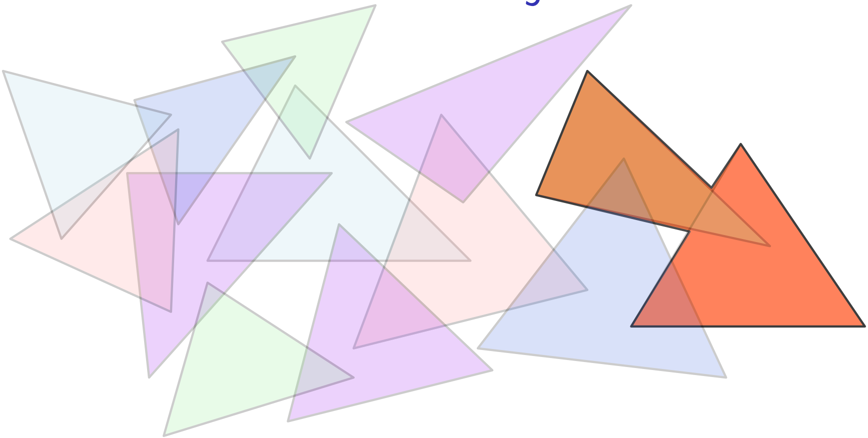
Minors of objects



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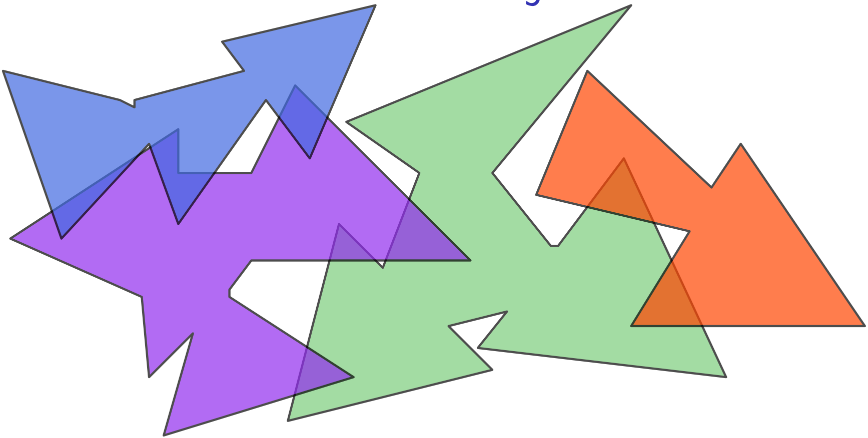
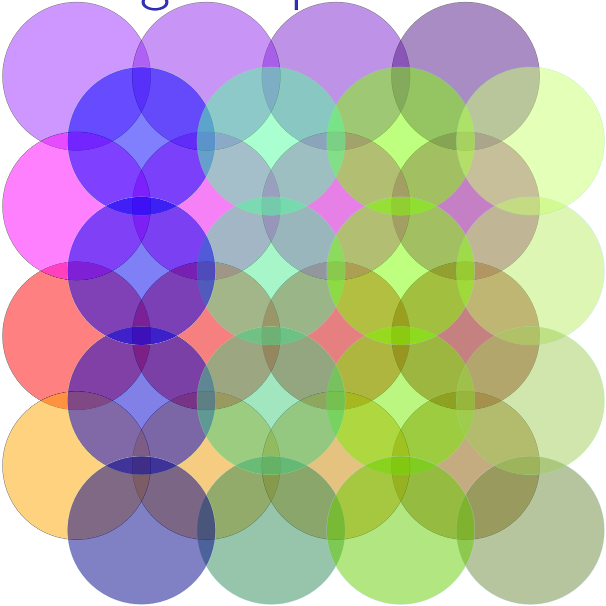
Minors of objects

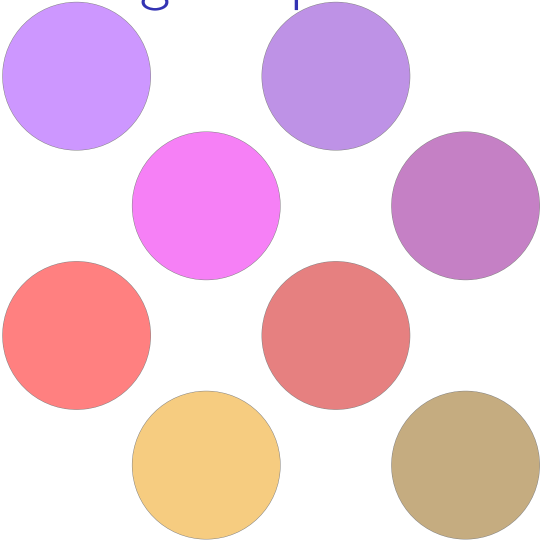


$\mathcal{G}$ is a **minor** of $\mathcal{F}$ if it can be obtained by deleting objects and taking unions of overlapping of objects.

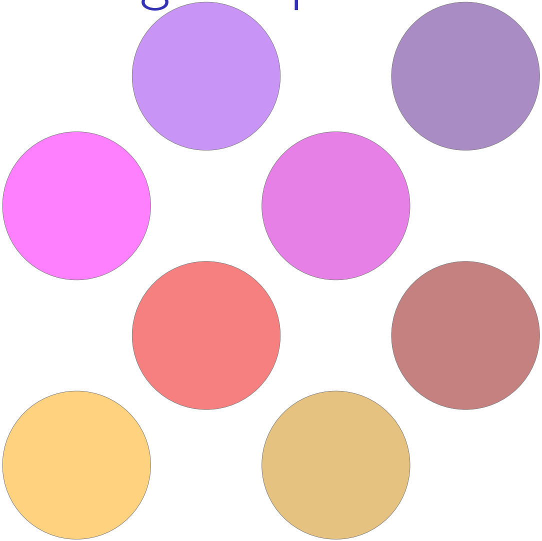
Large clique minors



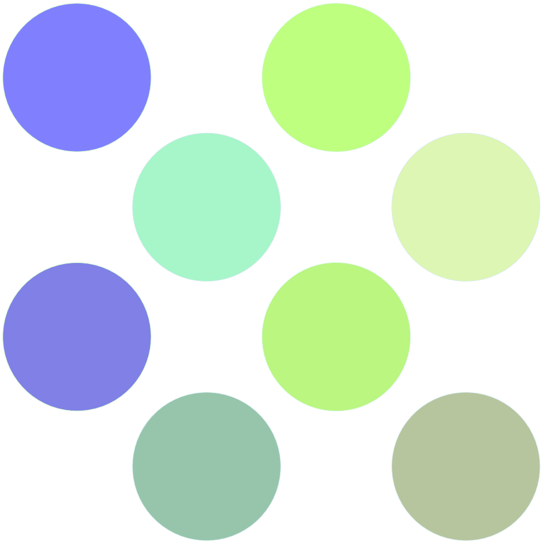
Large clique minors



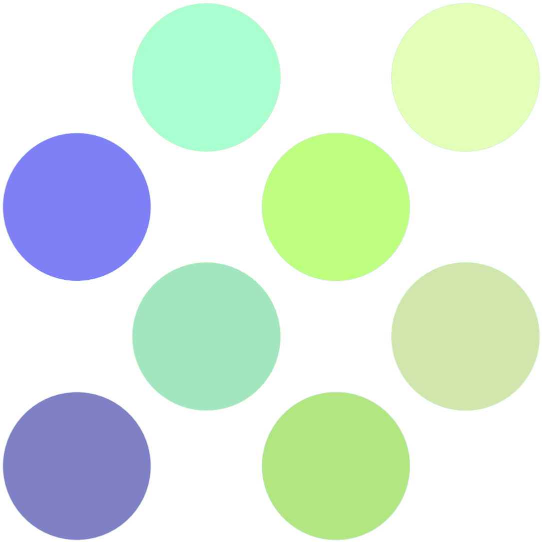
Large clique minors



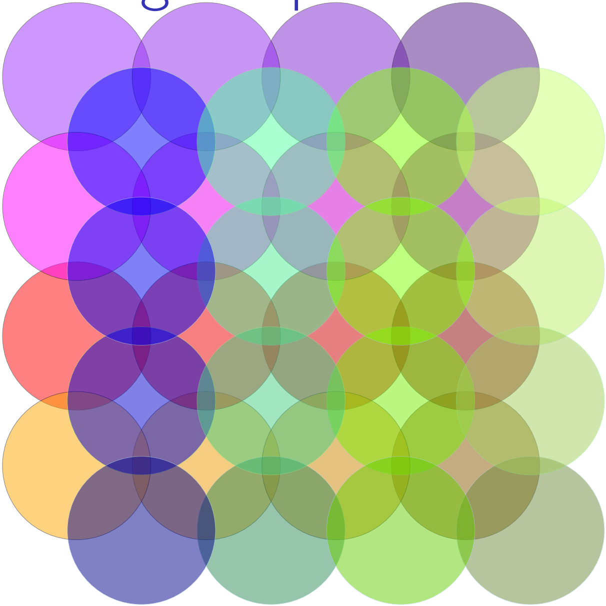
Large clique minors



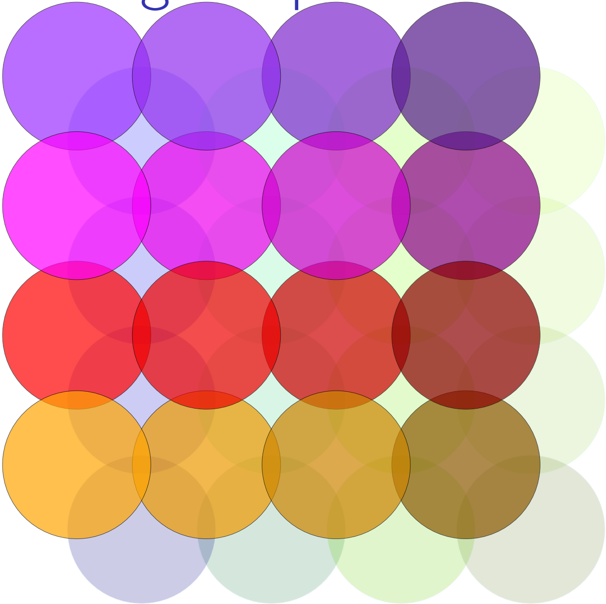
Large clique minors



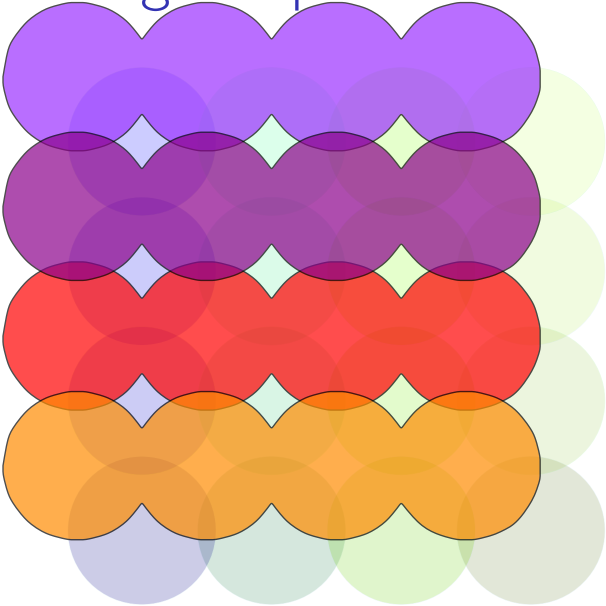
Large clique minors



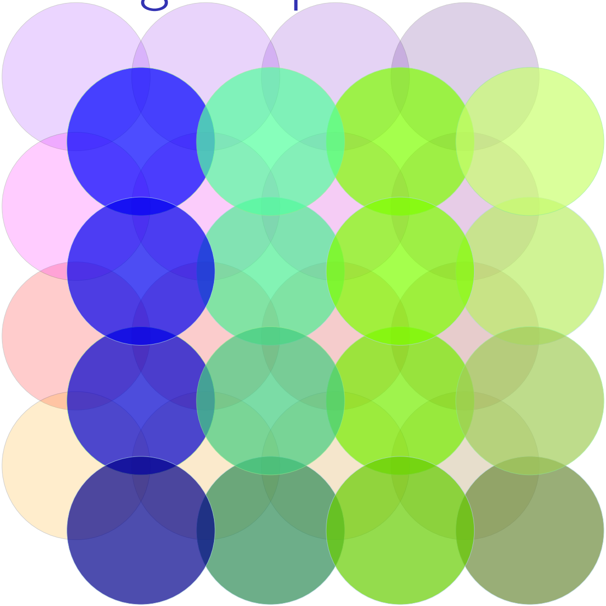
Large clique minors



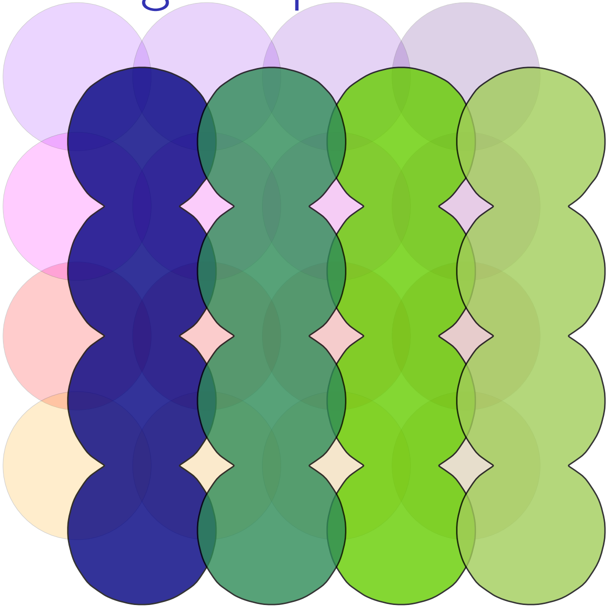
Large clique minors



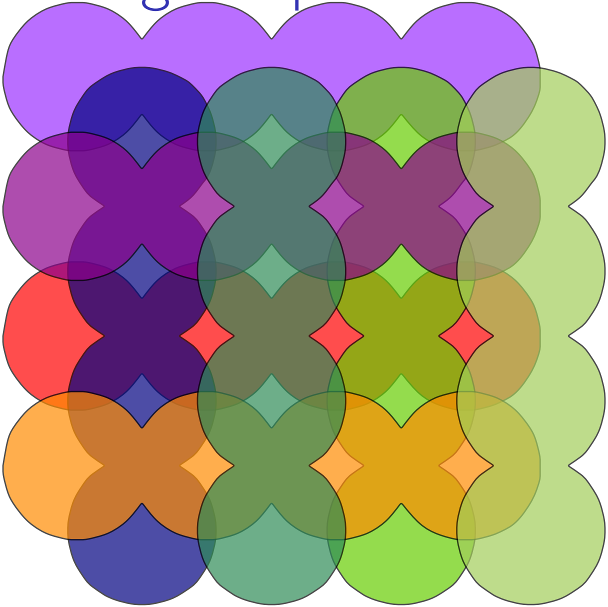
Large clique minors



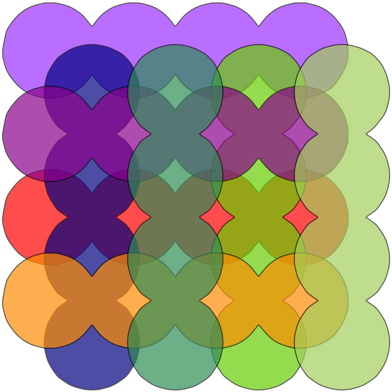
Large clique minors

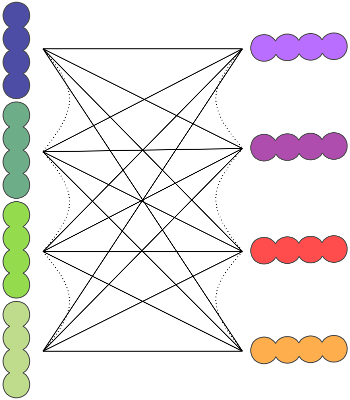


Large clique minors

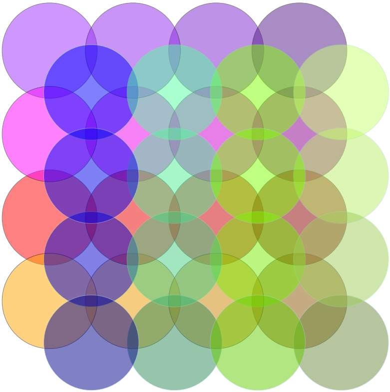


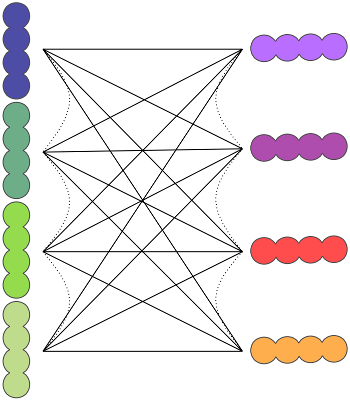
Large clique minors



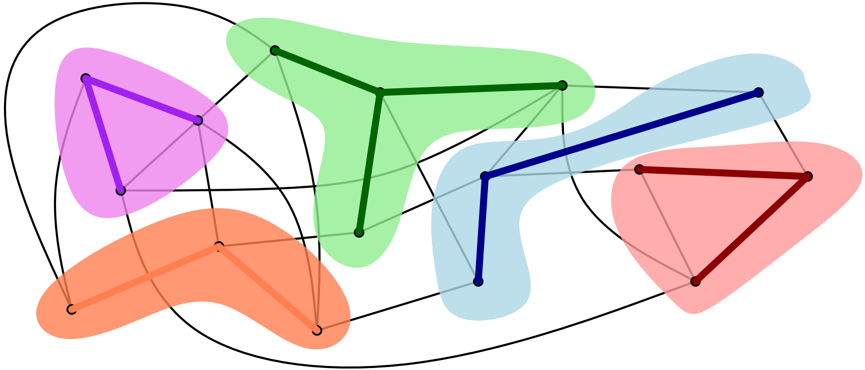


Large clique minors



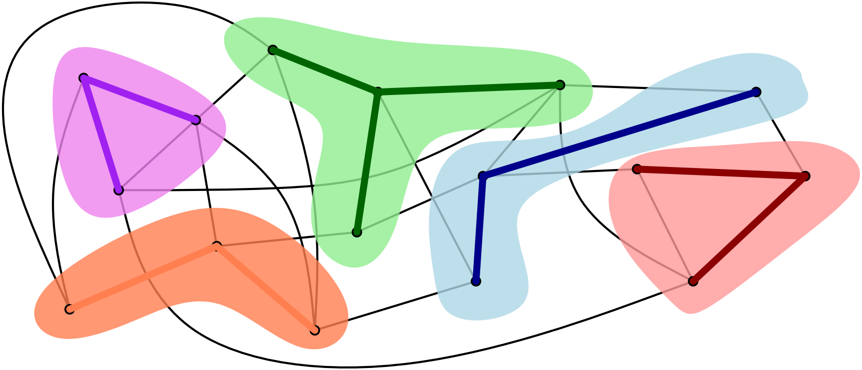


Shallow minors



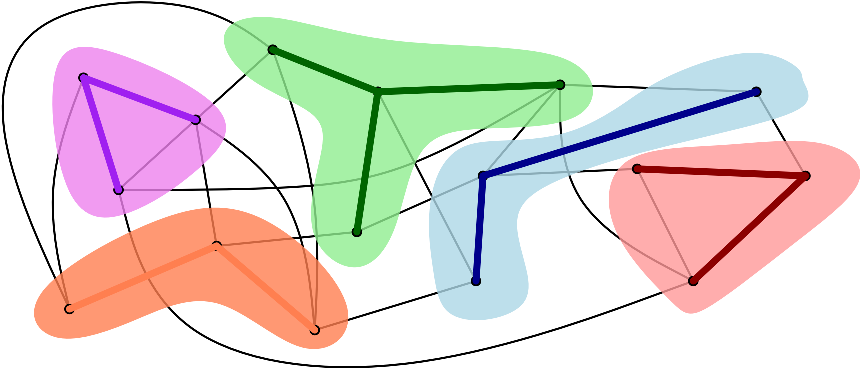
A vertex in H corresponds to a connected **cluster** of vertices in G.

Shallow minors



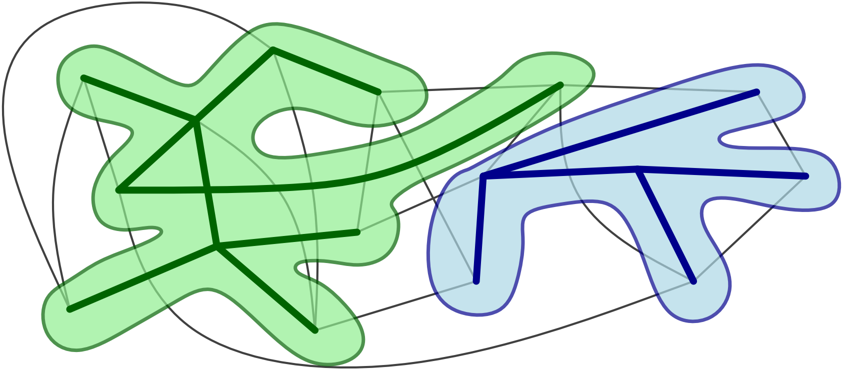
H is a **t-shallow minor** if each cluster induces a graph of radius t.

Shallow minors



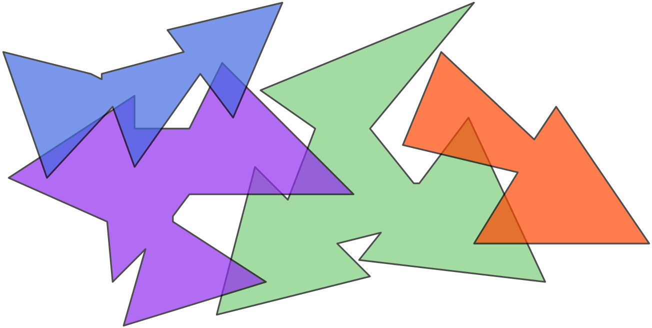
H is a **1-shallow minor** if each cluster induces a graph of radius 1.

Shallow minors



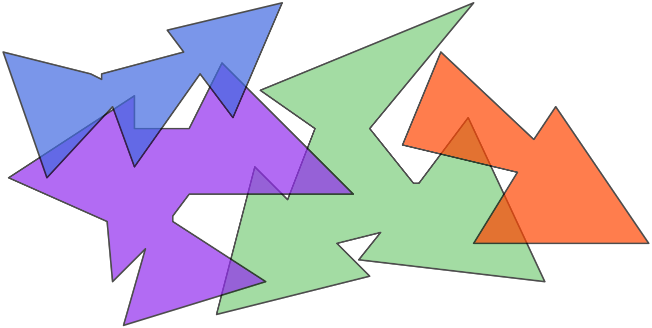
H is a **2-shallow minor** if each cluster induces a graph of radius 2.

Shallow minors of objects



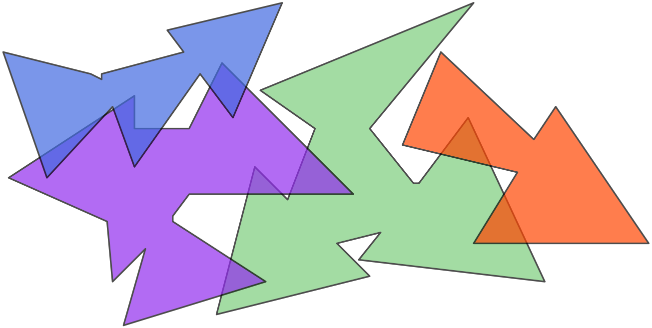
Each object in the minor corresponds to a **cluster** of objects in $\mathcal{F}$.

Shallow minors of objects



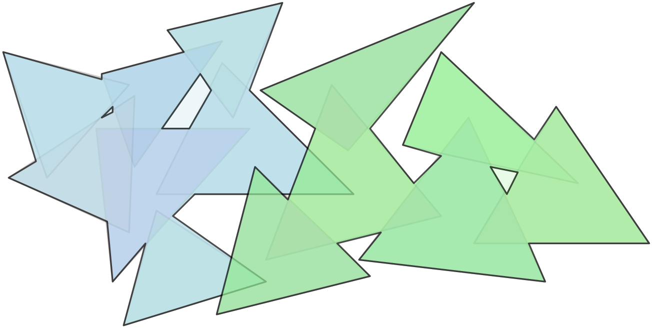
An object minor is a **t-shallow minor** if the intersection graph of each cluster has radius $\leq t$.

Shallow minors of objects



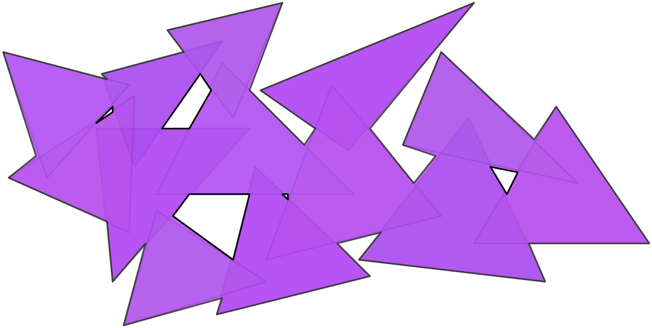
An object minor is a **1-shallow minor** if the intersection graph of each cluster has radius ≤ 1.

Shallow minors of objects



An object minor is a **2-shallow minor** if the intersection graph of each cluster has radius ≤ 2.

Shallow minors of objects

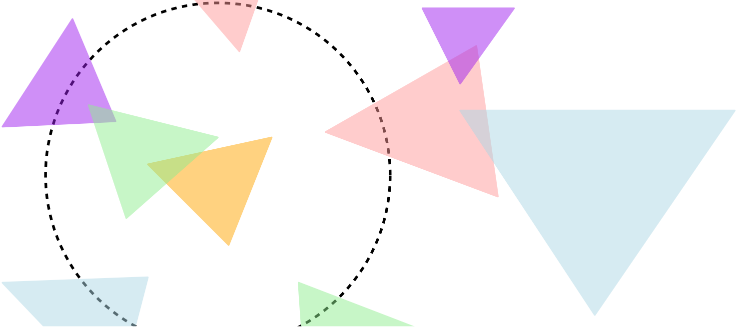


An object minor is a **3-shallow minor** if the intersection graph of each cluster has radius ≤ 3.

Shallow minors of low-density objects

A t-shallow minor of objects with density ρ has density $O(t^{O(d)}\rho)$

Shallow minors of low-density objects



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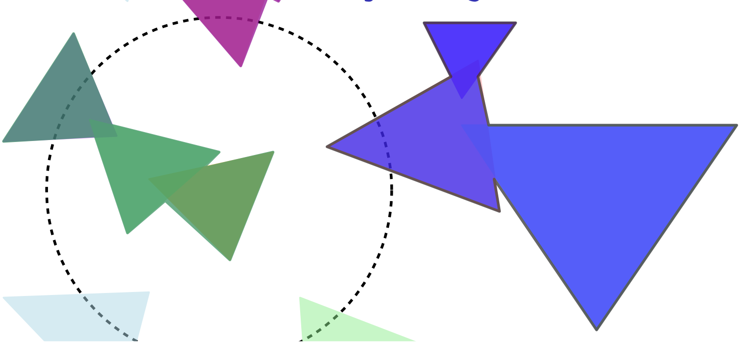
Shallow minors of low-density objects



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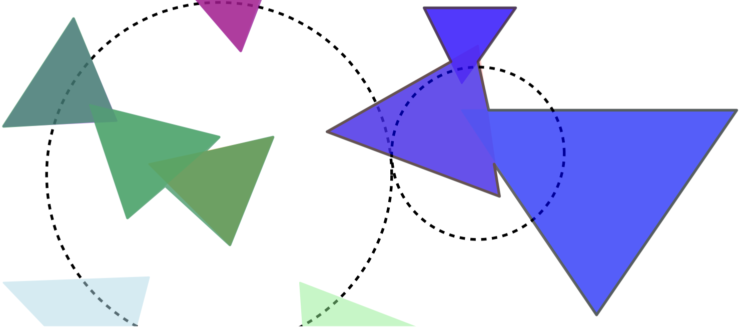
Shallow minors of low-density objects



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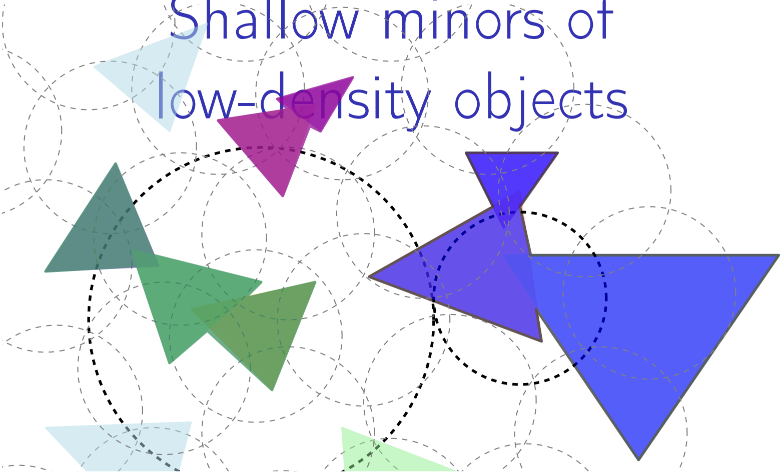
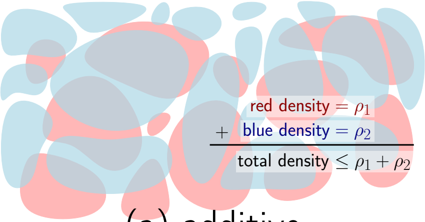
Shallow minors of low-density objects

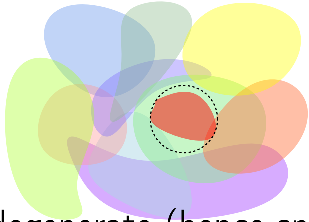


A t-shallow minor of objects with density ρ has density $O(t^{O(d)}\rho)$

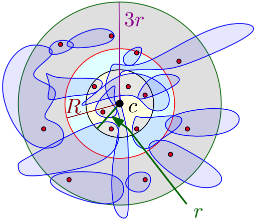
Recap: low-density graphs are...



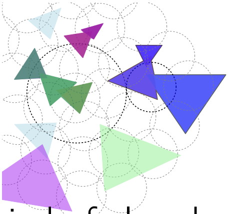
(a) additive



(b) degenerate (hence sparse)



(c) separable



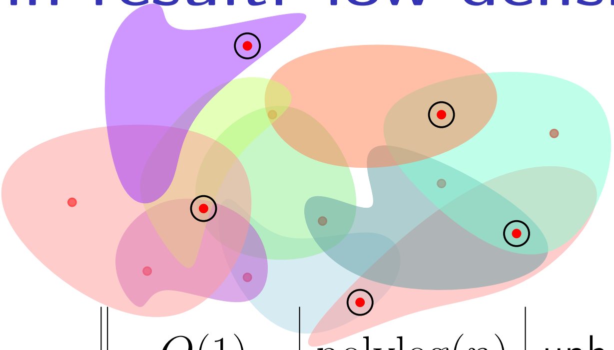
(d) kind of closed under shallow minors

Main result: low-density

$\rho = O(1)$: PTAS for hitting set, set cover, subset dominating set

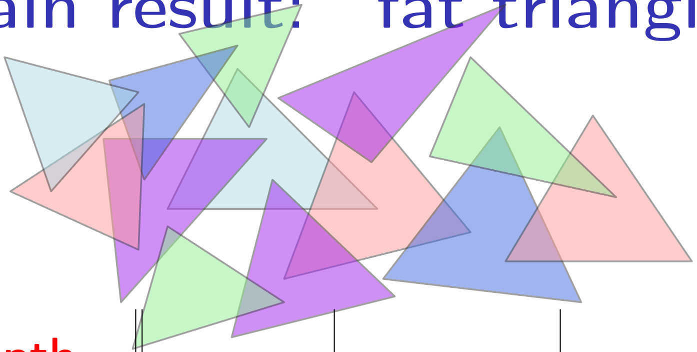
$\rho = \text{polylog}(n)$: QPTAS for same problems. No PTAS under ETH.

Main result: low-density

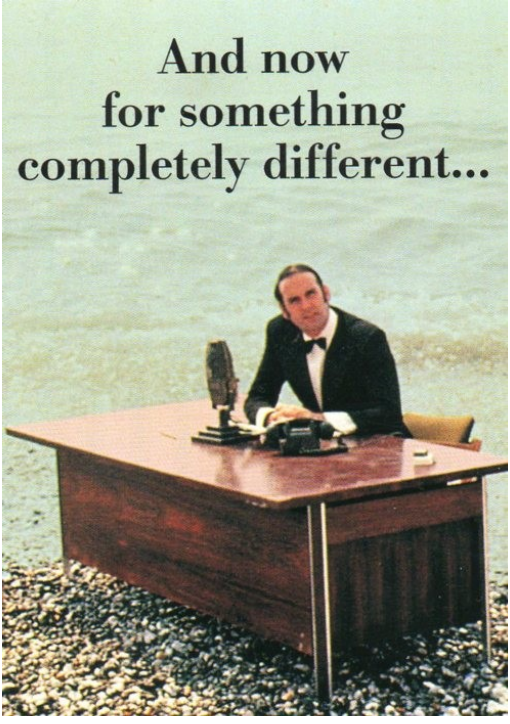


density ρ	$O(1)$	$\mathrm{polylog}(n)$	unbounded
hardness	NP-Hard	No PTAS	APX-Hard
algo	PTAS	QPTAS	

Main result: fat triangles

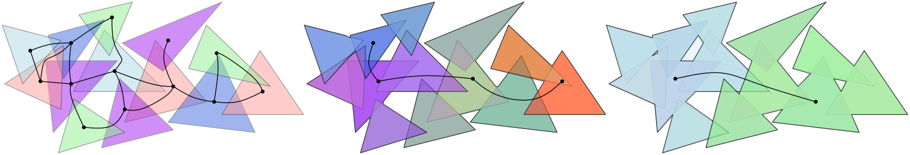


depth ~~density ρ~~	$O(1)$	$\mathrm{polylog}(n)$	unbounded
hardness	NP-Hard	No PTAS	APX-Hard
algo	PTAS	QPTAS	


And now
for something
completely different...

Shallow edge density

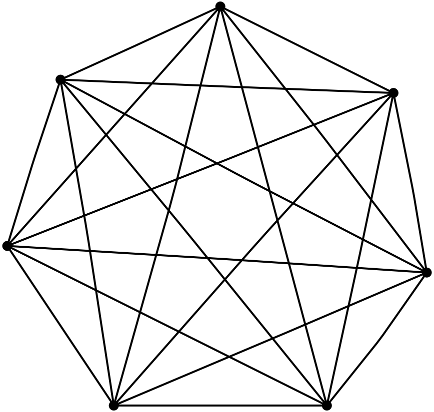
The r-**shallow density** of a graph is the max edge density over all r-shallow minors.



aka "greatest reduced average density"

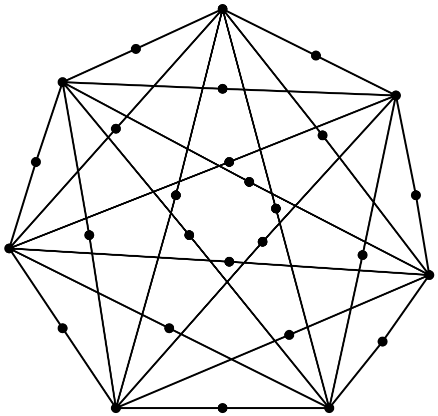
Nešetřil and Ossona de Mendez, 2008

Sparsity is not enough



Hide a clique by splitting the edges

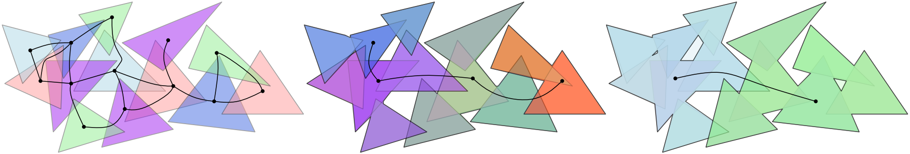
Sparsity is not enough



Hide a clique by splitting the edges

Expansion

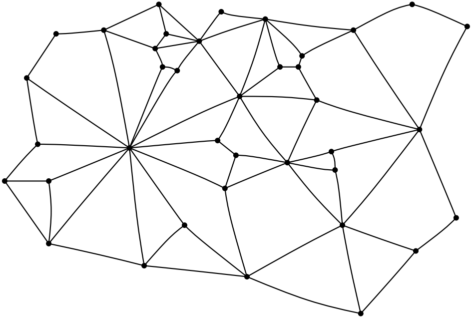
The **expansion** of a graph is the r-shallow density as a *function* of r.



e.g. *constant* expansion, *polynomial* expansion, *exponential* expansion

Nešetřil and Ossona de Mendez, 2008

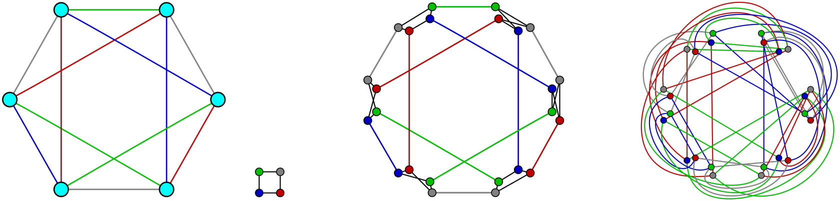
Examples of expansion



Planar graphs have constant expansion (Euler's formula)

Minor-closed classes have constant expansion

Sparsity is not enough



Constant degree expanders have exponential expansion

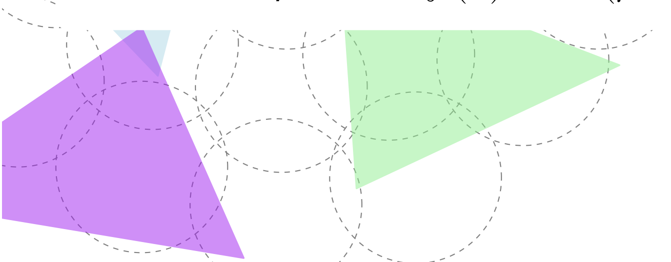
Wikipedia

Low density $\Rightarrow$ polynomial expansion

Graphs with density ρ have polynomial expansion $f(r) = O(\rho r^d)$

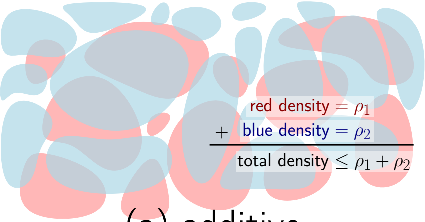
Low density ⇒ polynomial expansion

Graphs with density ρ have polynomial expansion $f(r) = O(\rho r^d)$

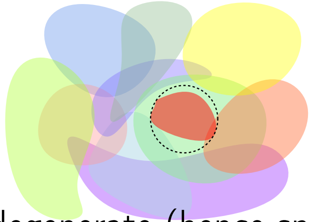


expansion	**examples**
constant	planar graphs, minor-closed families
polynomial	low-density graphs
exponential	expander graphs

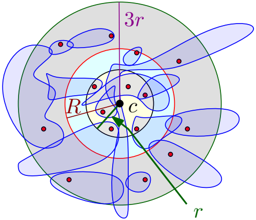
Recap: low-density graphs are...



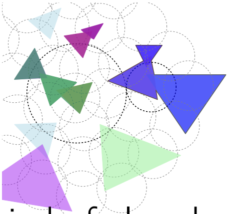
(a) additive



(b) degenerate (hence sparse)

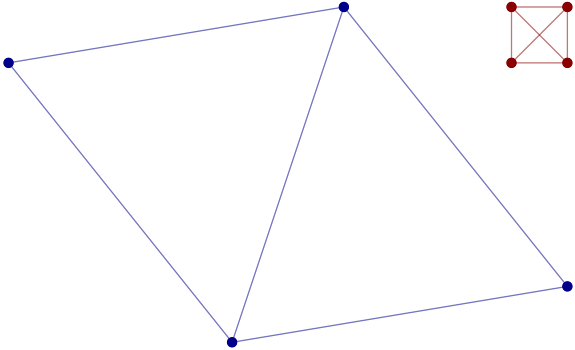


(c) separable



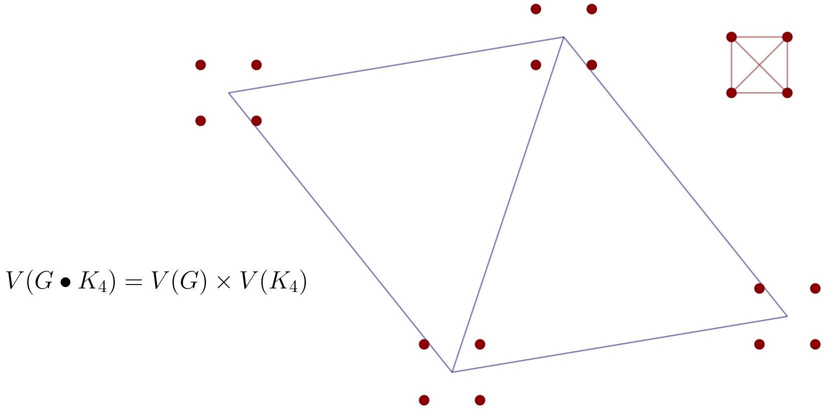
(d) kind of closed under shallow minors

Lexical product $G \bullet K_h$



The **lexical product** $G \bullet K_h$ blows up each vertex in G with the clique K_h.

Lexical product $G \bullet K_h$



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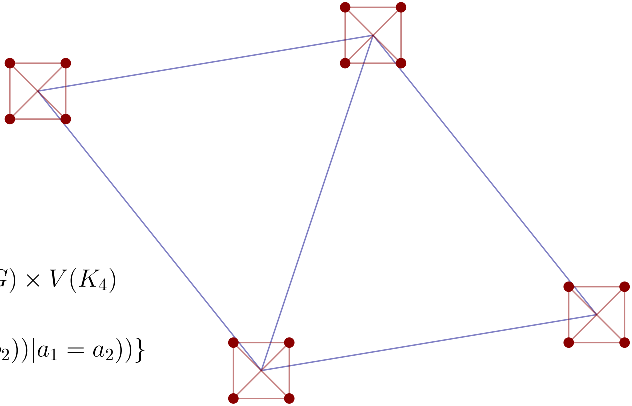
Lexical product $G \bullet K_h$

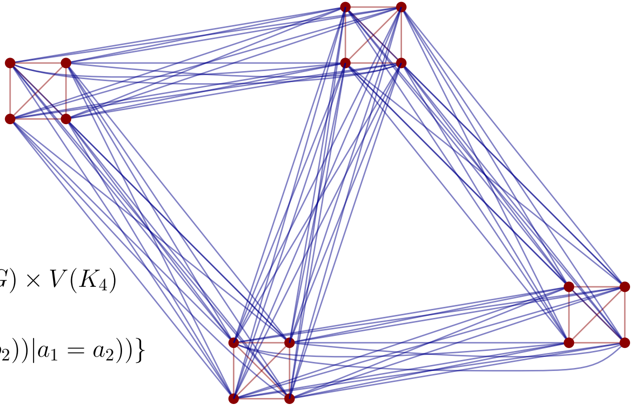


$V(G \bullet K_4) = V(G) \times V(K_4)$

$E(G \bullet K_4) =$
$\{((a_1, b_1), (a_2, b_2))|a_1 = a_2))\}$

The **lexical product** $G \bullet K_h$ blows up each vertex in G with the clique K_h.

Lexical product $G \bullet K_h$



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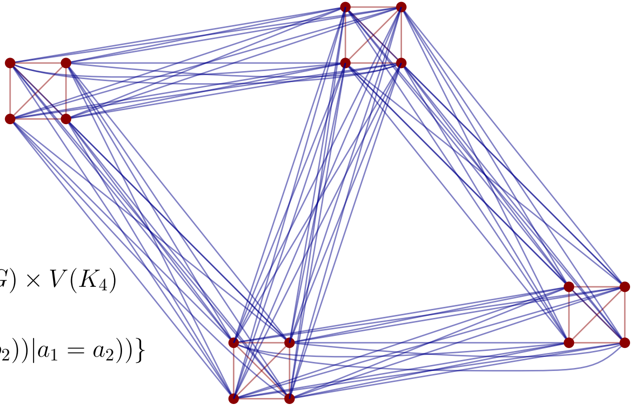

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If G has polynomial expansion, then $G \bullet K_h$ has polynomial expansion.

Small separators

Graphs with subexponential expansion have sublinear separators.

Nešetřil and Ossona de Mendez (2008)

Small separators

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Why?

Small separators

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1. No K_h as an ℓ-shallow minor implies a separator of size $O(n/\ell + 4\ell h^2 \log n)$.

Plotkin, Rao, and Smith (1994)

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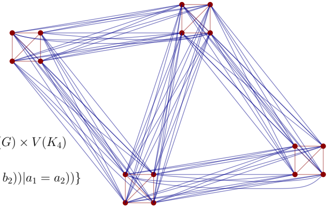
2. Small expansion => small clique minors as a function of depth

Shallow minors of polynomial expansion

Shallow minors of graphs with polynomial expansion have polynomial expansion.

(If H is an r_1-shallow minor of G, then an r_2-shallow minor of H is an $(r_1 \cdot r_2)$-shallow minor of G, and $\mathrm{poly}(r_1 \cdot r_2) = \mathrm{poly}(r_2)$.)

Recap: polynomial expansion graphs are...



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(a) closed under lexical product

FREE

(b) degenerate

via separator for excluded shallow minors

FREE

(c) separable

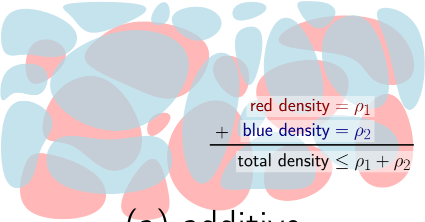
(d) kind of closed under shallow minors

Main result: polynomial expansion

- Graph G with polynomial expansion
- *PTAS for (subset) dominating set*
- Extensions: multiple demands, reach, connected dominating set, vertex cover.

PTAS for independent set

Recap: low-density graphs are...



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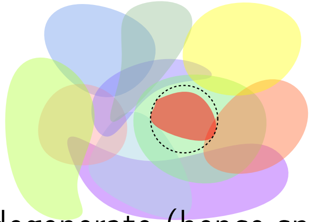

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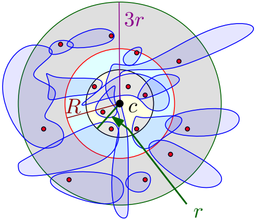

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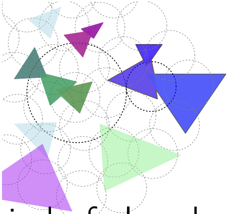
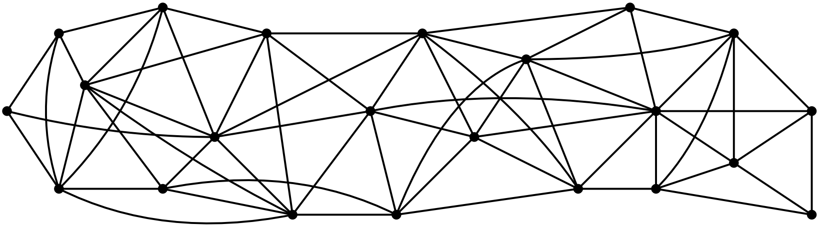


(d) kind of closed under shallow minors

Balanced separators



Input: $G = (V, E)$ with n vertices.

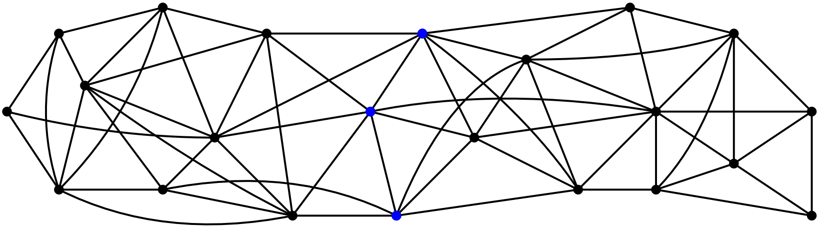
Output: Partition $V = X \sqcup S \sqcup Y$ s.t.

(a) $|X|, |Y| \leq .99n$.

(b) $|S| \leq n^{.99}$.

(c) No edges run between X and Y.

Balanced separators



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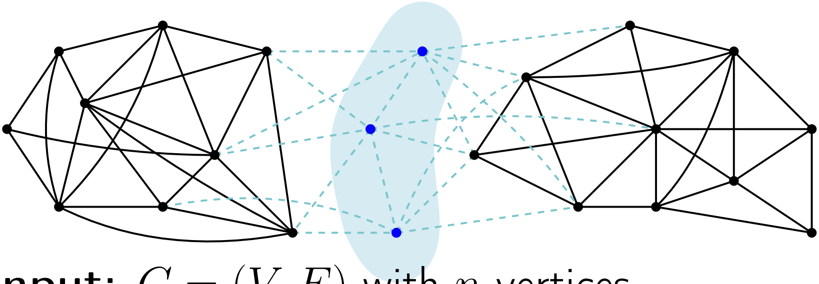

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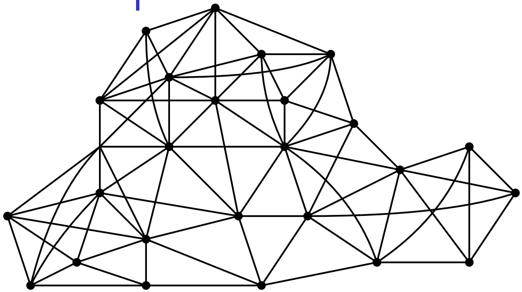
Well-separated divisions



Input: $G = (V, E)$ w/ n vertices, $\epsilon \in (0, 1)$

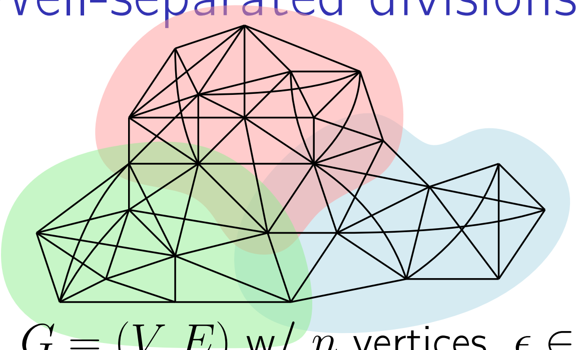
Output: Cover $V = C_1 \cup C_2 \cup \cdots \cup C_m$ s.t.

(a) $|C_i| = \mathrm{poly}(1/\epsilon)$ for all i

(b) No edges between $C_i \setminus C_j$ and $C_j \setminus C_i$

(c) $\sum_i |C_i| \leq (1 + \epsilon)n$

Well-separated divisions



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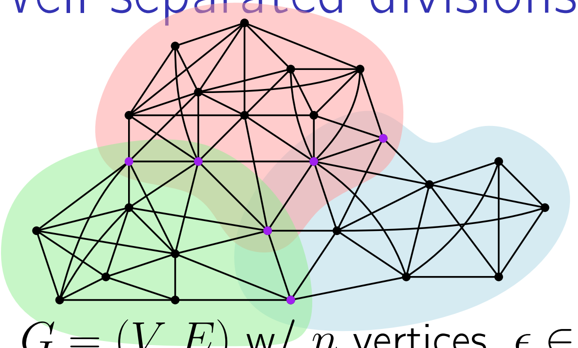

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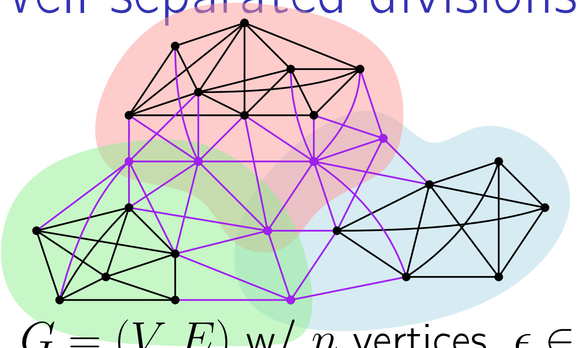

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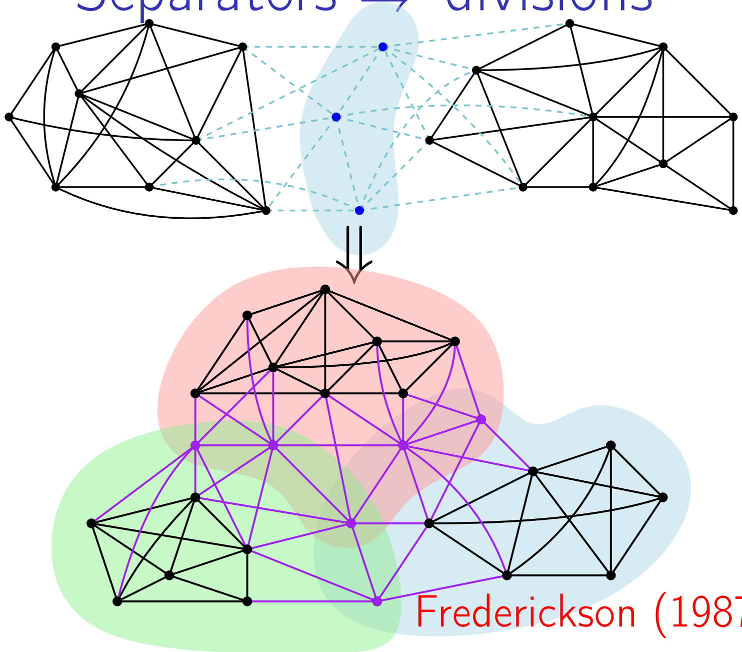
Separators ⇒ divisions



Frederickson (1987)

Local search

```
local-search(G = (V, E), ε)
  L ← ∅, λ ← poly(1/ε)
  while there exists S ⊆ V s.t.
      (a) |S| ≤ λ
      (b) L△S is an independent set
      (c) |L△S| > |L|
    do L ← L△S
  end while
  return L
```

Independent set: proof setup

O: Optimal solution

L: λ-locally optimal sol'n for $\lambda = \mathrm{poly}(1/\epsilon)$

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PTAS for dominating set

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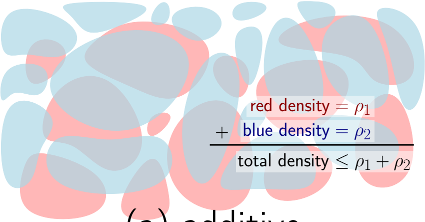

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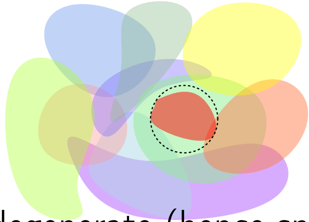

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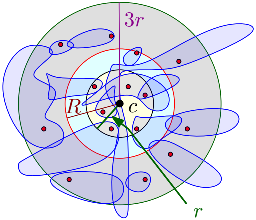
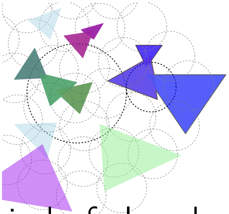


(c) separable



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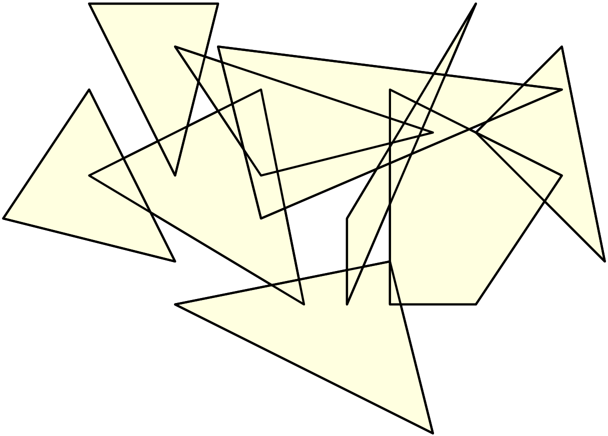
Flowers



Glue an object in the dominating set with the neighboring objects it dominates.

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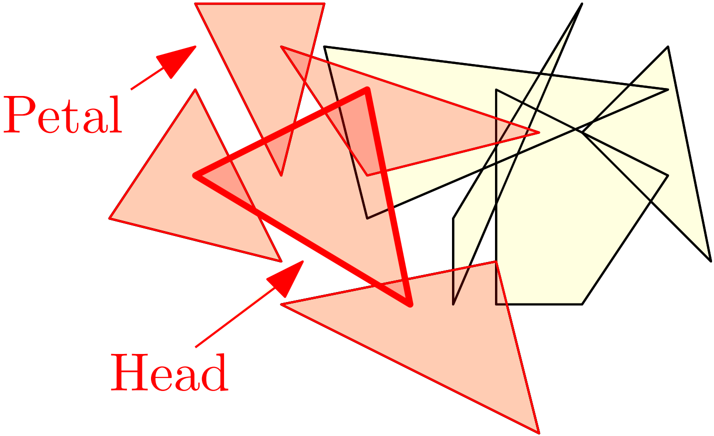

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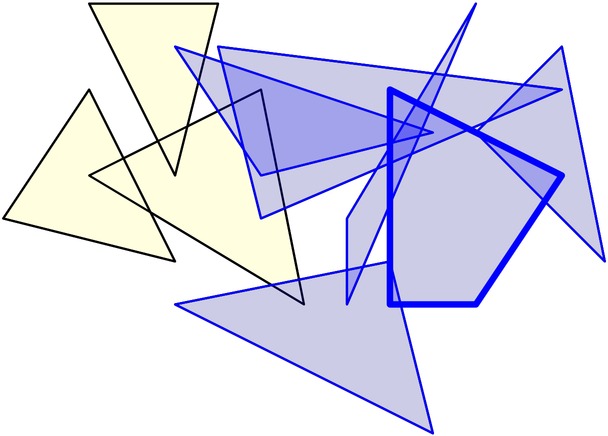

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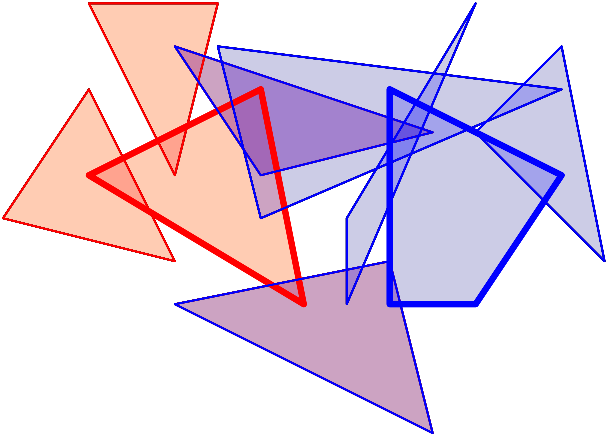

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O = optimal solution, $\tilde{O}$ = flowers of O

L = locally-optimal sol'n, $\tilde{L}$ = flowers of L

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$\blacksquare$

thanks!

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J. Nešetřil and P. Ossona de Mendez. Grad and classes with bounded expansion ii. algorithmic aspects. *European J. Combin.*, 29(3):777–791, 2008.

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