A Fast Approximation for Maximum Weight Matroid Intersection

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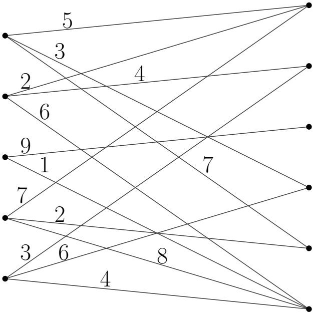
University of Illinois at Urbana-Champaign

UIUC Theory Seminar
February 8, 2016

Max. weight bipartite matching



Input:

bipartite graph $G = (L \sqcup R, E)$

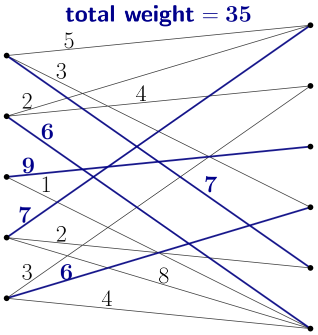
weights $w : E \to \mathbb{R}_{\geq 0}$

Output:

matching $M \subseteq E$ maximizing

$$w(M) \stackrel{\text{def}}{=} \sum_{e \in M} w(e)$$

Max. weight bipartite matching



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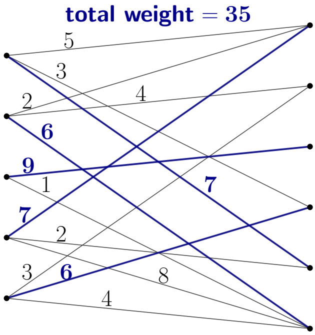

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Approximate output:

matching $M \subseteq E$ s.t. $w(M) \geq (1 - \epsilon)\mathrm{OPT}$

Fast approximations: *bipartite matching*

	exact		approximate
cardinality	$O(m\sqrt{n})$	Hopcroft and Karp [1973]	$O(m/\epsilon)$
	$\widetilde{O}(m^{10/7})$	Mądry [2013]	
weighted	$O(mn + n^2 \log n)$	Fredman and Tarjan [1987]	$O(m \log(1/\epsilon)/\epsilon)$
	$O(m\sqrt{n} \log W)$	Duan and Su [2012]	Duan and Pettie [2014]

(m = edges, n = vertices, W = max weight)

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(m = edges, n = vertices, W = max weight)

(!) For fixed ϵ, weighted approximation is faster than unweighted exact

Duan and Pettie [2014]

(extends to general matching)

1. Primal-dual
   - *Only approximates dual optimal conditions*
2. Scaling reduces weighted problem to unweighted
3. Runs an approximate subroutine at each scale
4. Updates dual variables with small loss from approximation

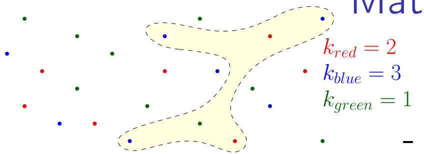
Matroids

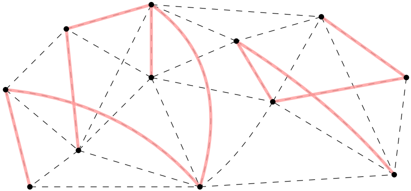
$$\mathcal{M} = (\mathcal{N}, \mathcal{I})$$

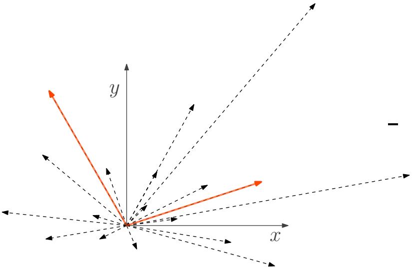
$\mathcal{N}$: **ground set** of elements

$\mathcal{I} \subseteq 2^{\mathcal{N}}$: **independent** (feasible) sets

Matroids







- Empty set is independent
- Subsets of independent sets are independent
- Maximal independent sets have same cardinality

max independent set is a ***base***

max cardinality is the ***rank***

- If $A, B \in \mathcal{I}$ and $|A| < |B|$, then there is $b \in B \setminus A$ s.t. $A + b \in \mathcal{I}$

Matroid Intersection

(same ground set)

$$\mathcal{M}_1 = (\mathcal{N}, \mathcal{I}_1) \qquad \mathcal{M}_2 = (\mathcal{N}, \mathcal{I}_2)$$

$$\mathcal{M}_1 \cap \mathcal{M}_2 \triangleq (\mathcal{N}, \mathcal{I}_1 \cap \mathcal{I}_2)$$

e.g. bipartite matchings, arborescences

Matroid intersection problems

Maximum cardinality matroid intersection

Input: matroids $\mathcal{M}_1 = (\mathcal{N}, \mathcal{I}_1)$, $\mathcal{M}_2 = (\mathcal{N}, \mathcal{I}_2)$

Output: $S \in \mathcal{I}_1 \cap \mathcal{I}_2$ maximizing $|S|$

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Maximum weight matroid intersection

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Oracle Model

Independence queries of the form "Is $S \in \mathcal{I}$?"

Running times for matroid intersection

Running times for matroid intersection

Cardinality

$O(nk^{1.5}Q)$ Cunningham [1986]

($n = |\mathcal{N}|$, $k = \operatorname{rank}(\mathcal{M}_1 \cap \mathcal{M}_2)$, $Q =$ cost of indep. query)

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Weighted

($W = \max\{w(e) : e \in \mathcal{N}\}$)

$O(nk^2Q)$ Frank [1981], Brezovec et al. [1986], Schrijver [2003]

$O(n^2\sqrt{k}\log(kW)Q)$ Fujishige and Zhang [1995]

$O(nk^{1.5}WQ)$ Huang et al. [2014]

$O((n^2\log(n)Q + n^3\,\mathrm{polylog}(n))\log(nW))$

Lee et al. [2015]

Approximate matroid intersection

Input: $\mathcal{M}_1 = (\mathcal{N}, \mathcal{I}_1)$, $\mathcal{M}_2 = (\mathcal{N}, \mathcal{I}_2)$,
$w : \mathcal{N} \to \mathbb{R}_{\geq 0}$, $\epsilon > 0$

Output: $S \in \mathcal{I}_1 \cap \mathcal{I}_2$ s.t. $w(S) \geq (1 - \epsilon)\mathrm{OPT}$
$(\mathrm{OPT} = \max\{w(T) : T \in \mathcal{I}_1 \cap \mathcal{I}_2\})$

Previous bound:

$O(nk^{1.5} \log(k) Q/\epsilon)$ Huang et al. [2014]

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Main result: $(1 - \epsilon)$-approximation in time

$$O(nkQ \log^2(\epsilon)/\epsilon^2)$$

Fast approximations: *matroid intersection*

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A Fast Approximation for Maximum Weight Bipartite Matching

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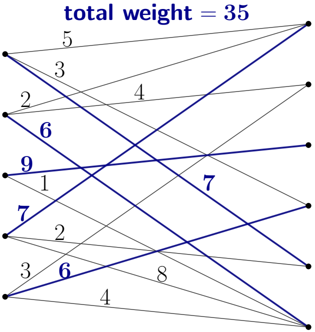

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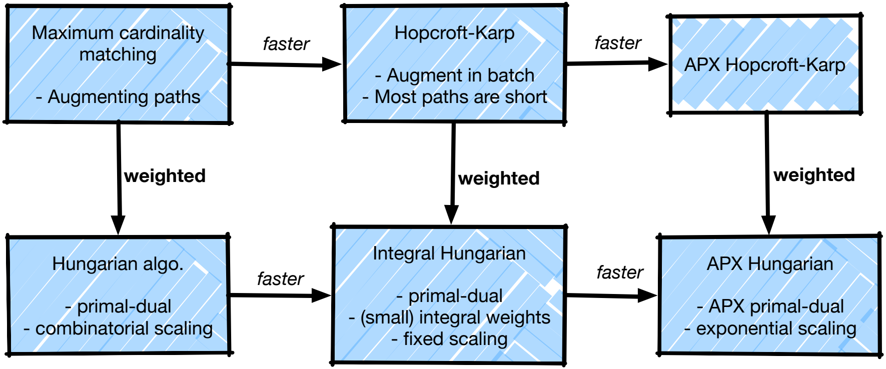
Maximum cardinality matching
- Augmenting paths
faster
Hopcroft-Karp
- Augment in batch
- Most paths are short
faster
APX Hopcroft-Karp
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Hungarian algo.
- primal-dual
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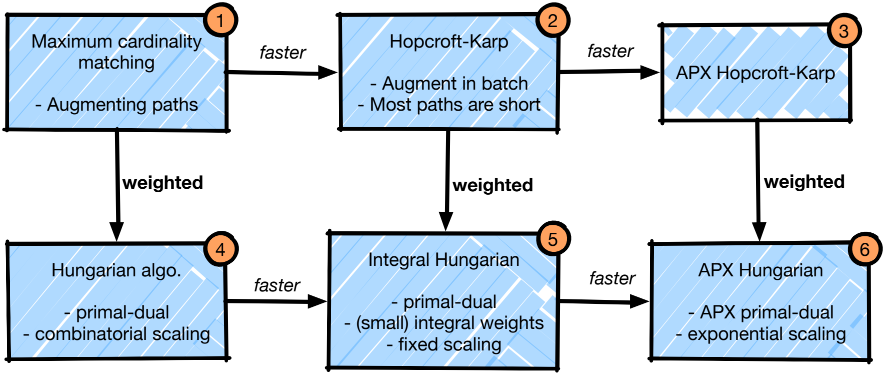
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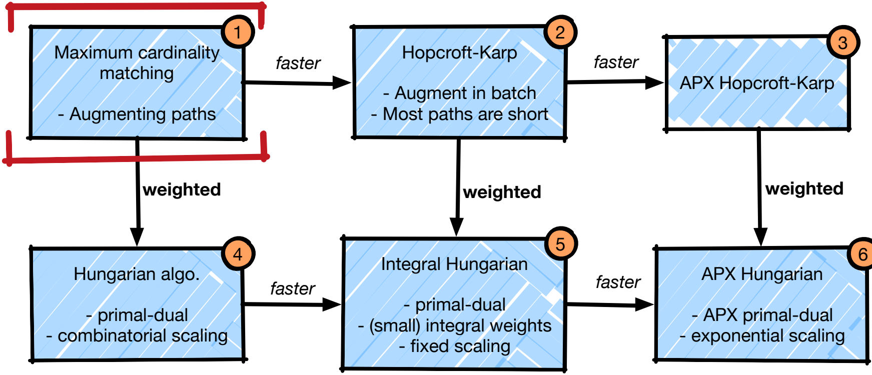
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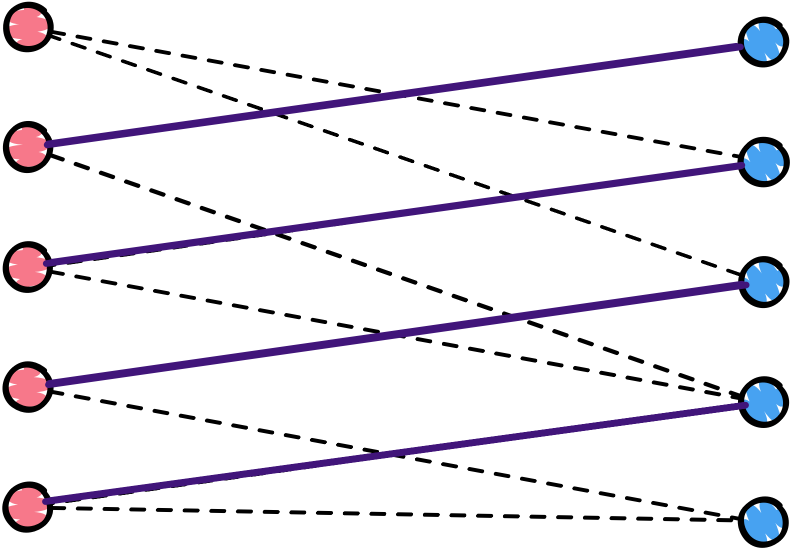

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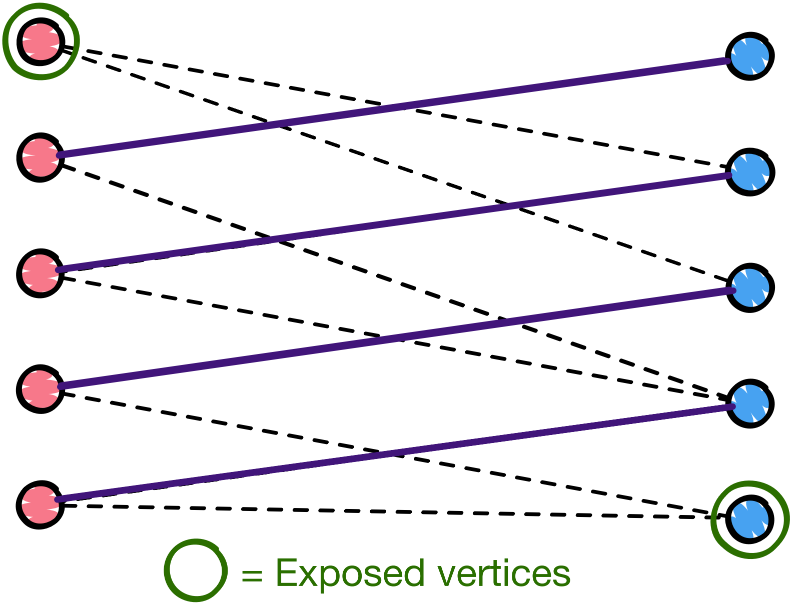

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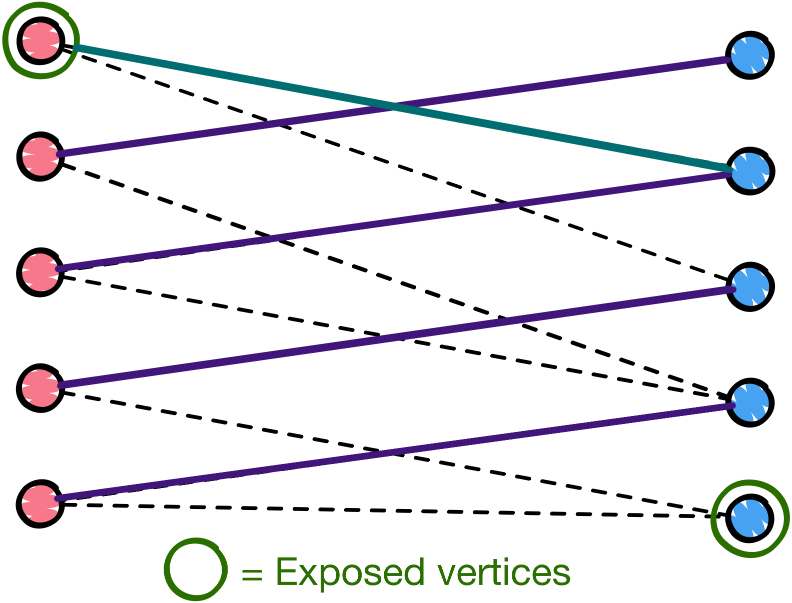

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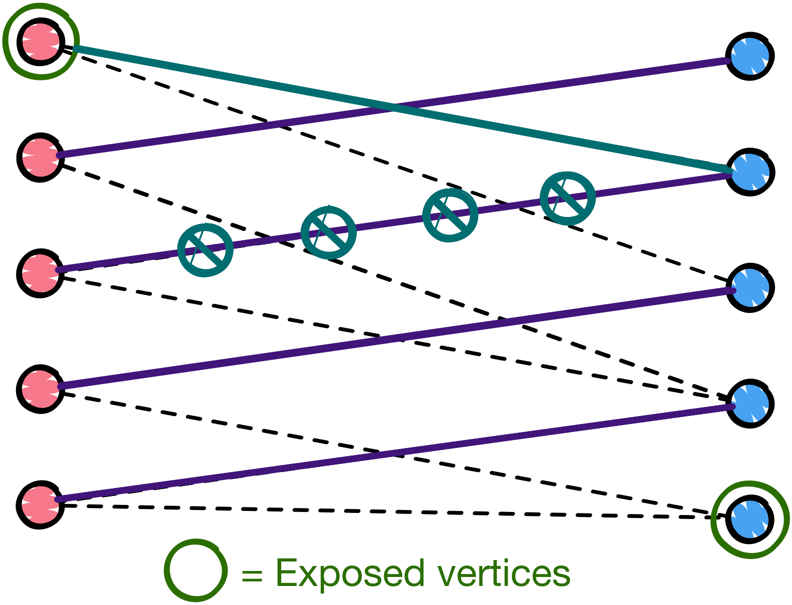

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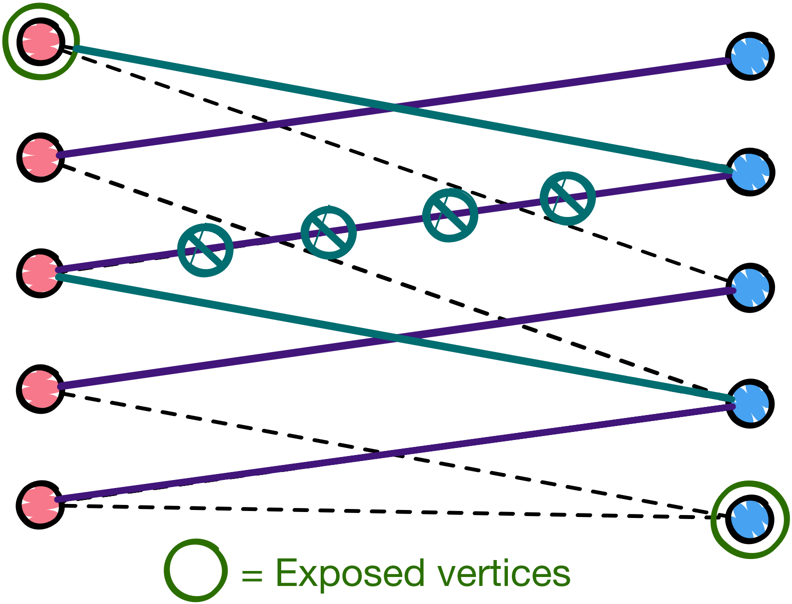

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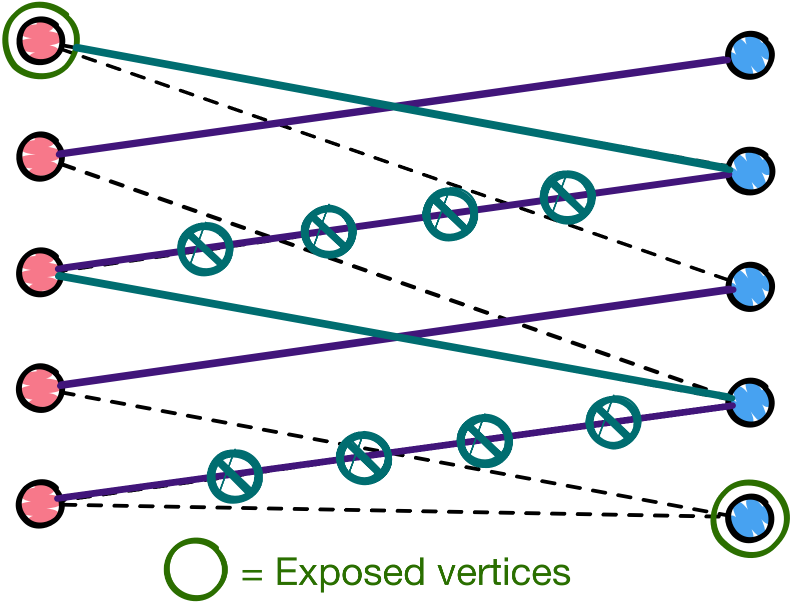
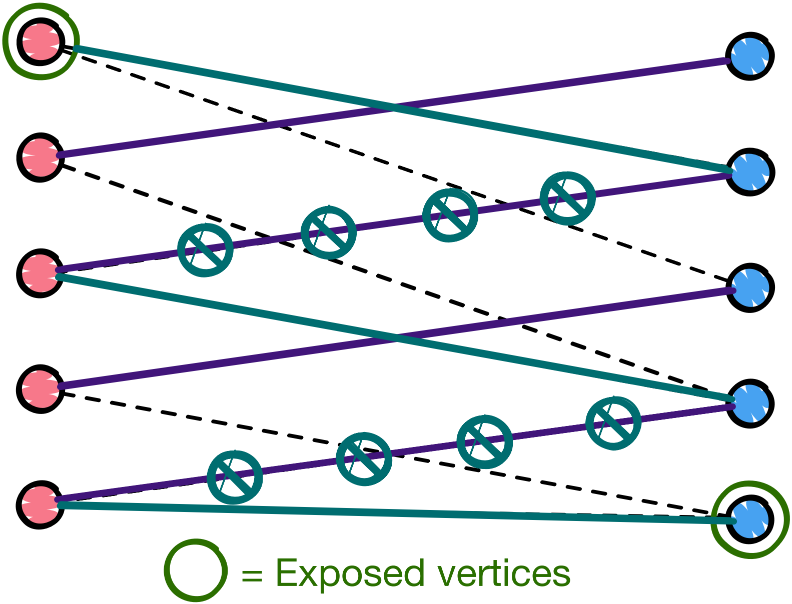


= Exposed vertices

Augmenting paths



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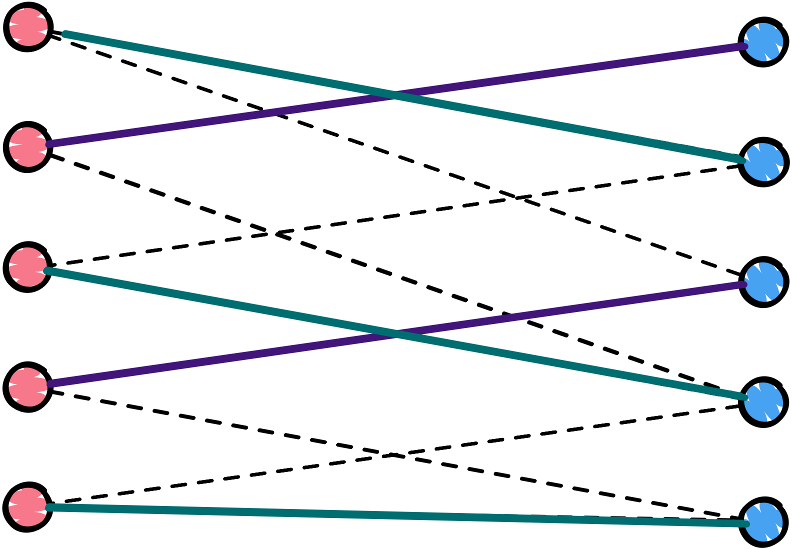
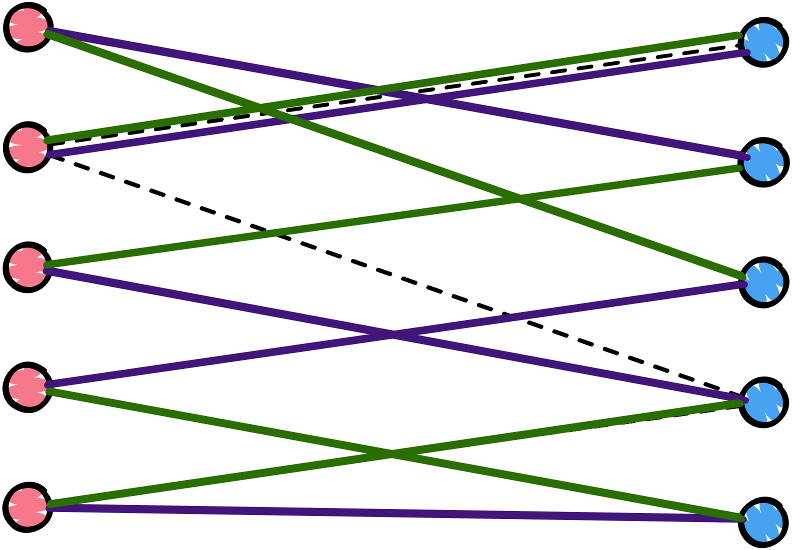


Max. matching ⇔ no aug. paths



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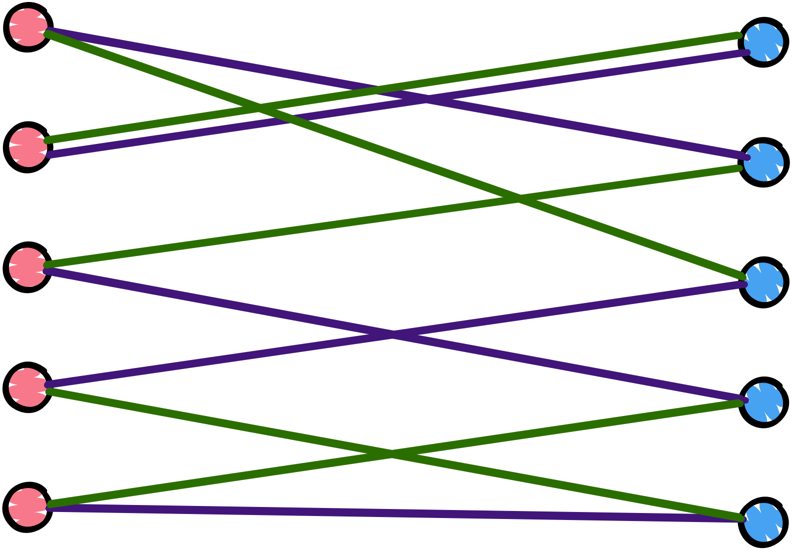

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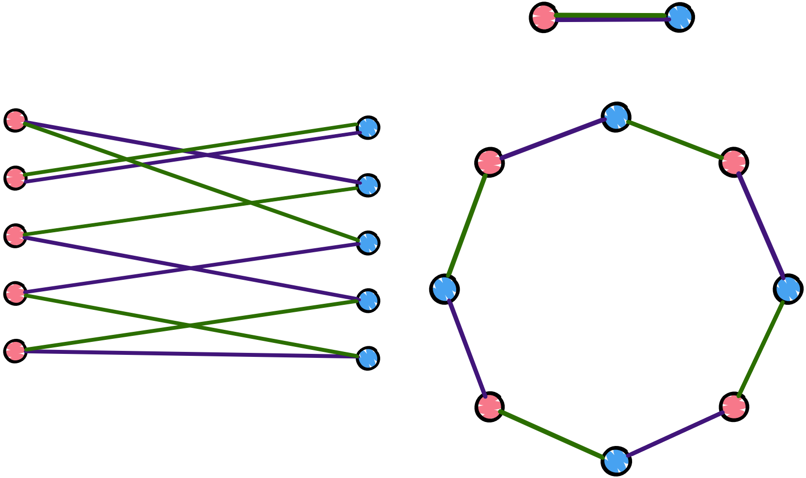
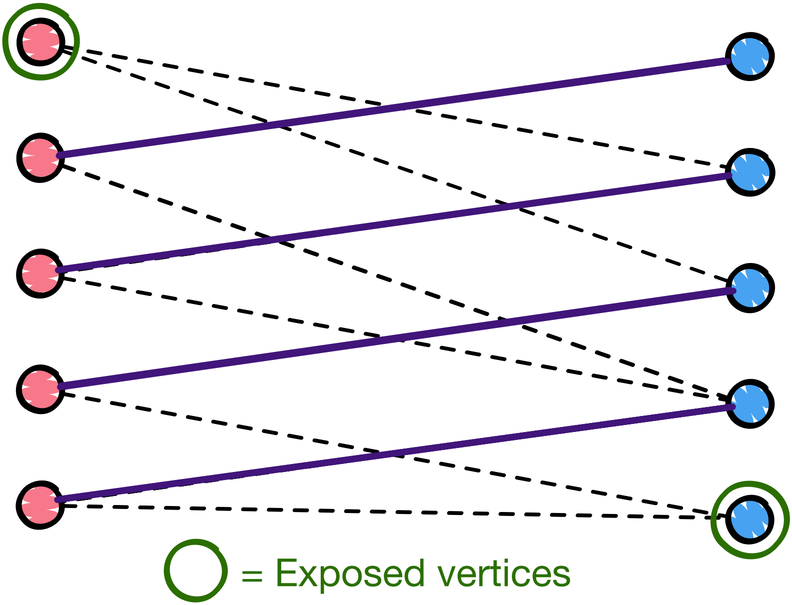


Finding augmenting paths



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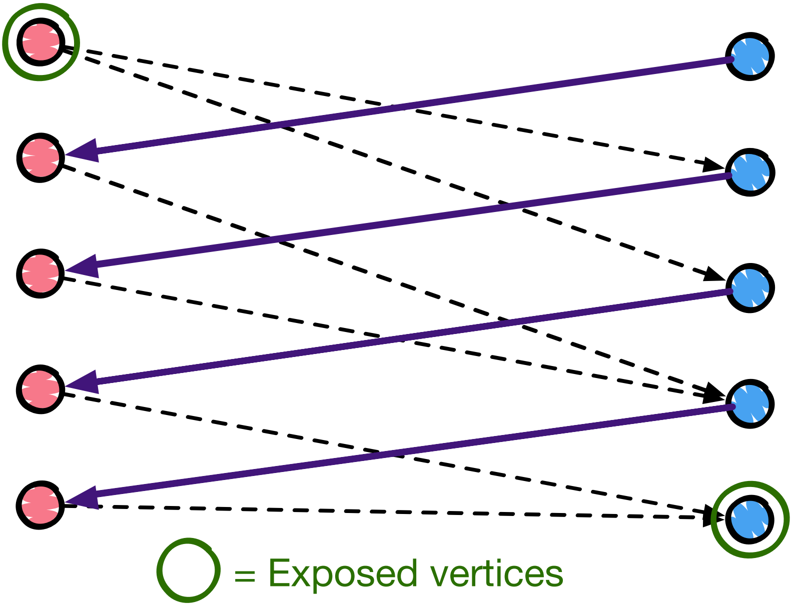

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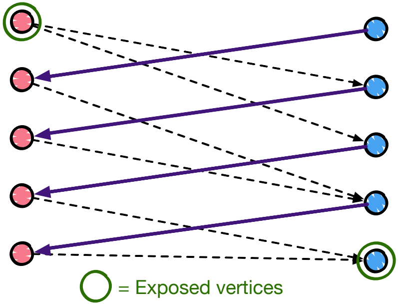

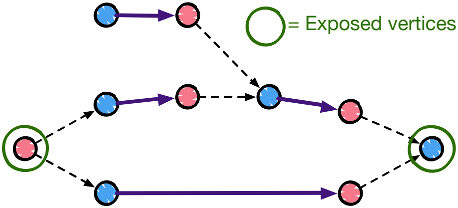

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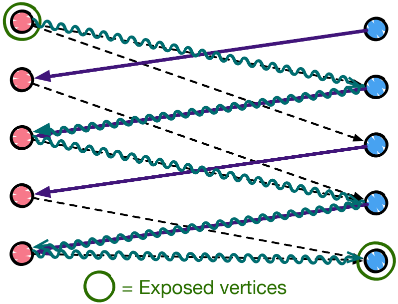
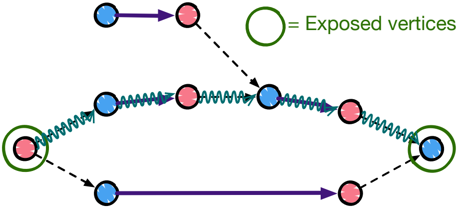
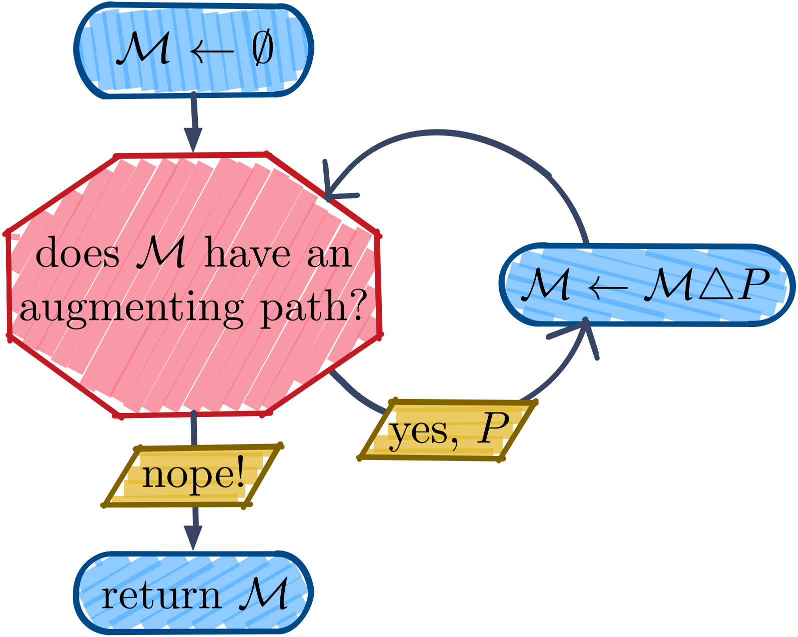
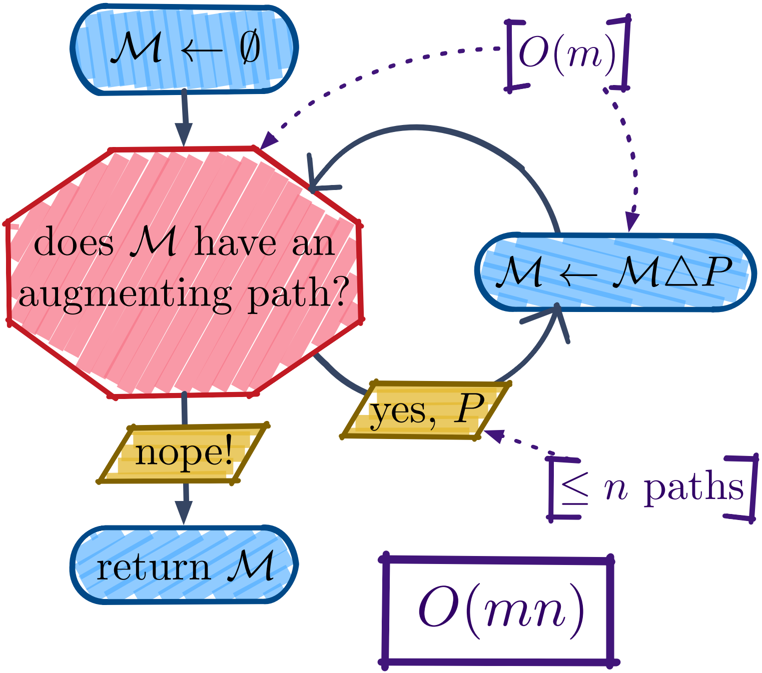





$\mathcal{M} \leftarrow \emptyset$
does $\mathcal{M}$ have an augmenting path?
$\mathcal{M} \leftarrow \mathcal{M} \triangle P$
yes, P
nope!
return $\mathcal{M}$



$\mathcal{M} \leftarrow \emptyset$
$O(m)$
does $\mathcal{M}$ have an augmenting path?
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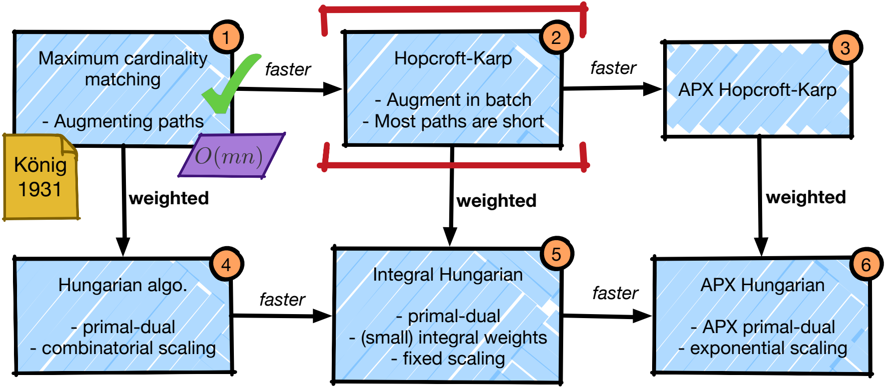
yes, P
nope!
$\leq n$ paths
return $\mathcal{M}$
$O(mn)$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König 1931
$O(mn)$
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Hopcroft-Karp

Most augmenting paths are short

Augment in batch and increase shortest aug. path length

Matching $\mathcal{M}$, $k \stackrel{\text{def}}{=} |\mathcal{M}| < \text{OPT}$

Lemma. *$\mathcal{M}$ has aug. path P of length*

$$|P| \leq 2\left\lfloor \frac{k}{\text{OPT} - k} \right\rfloor + 1$$

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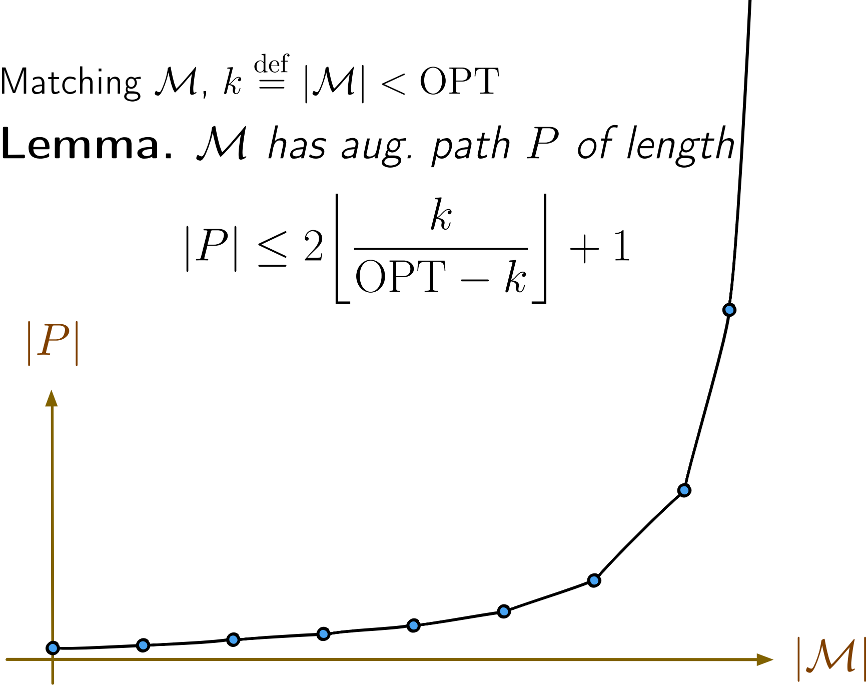

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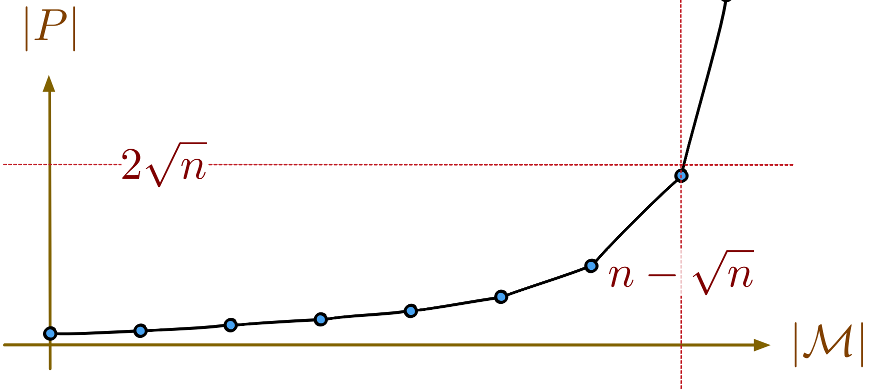

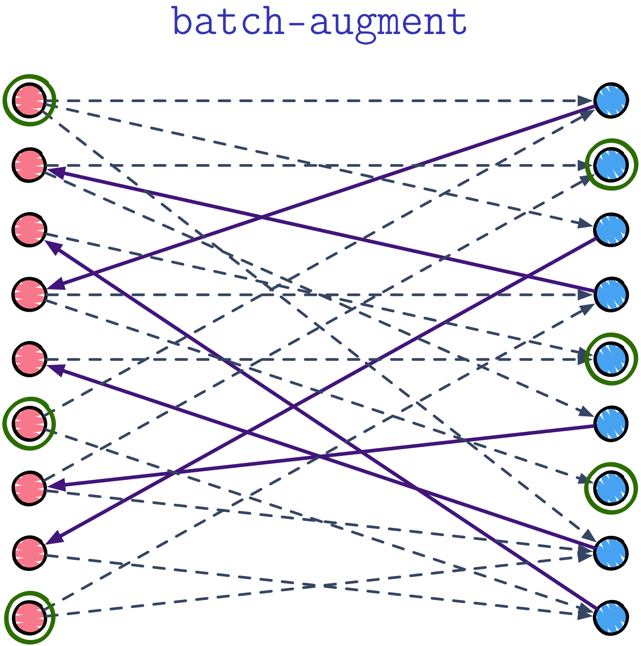
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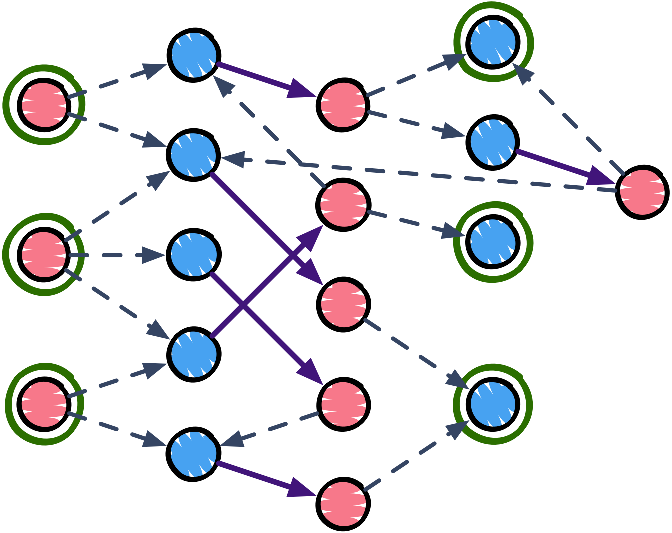
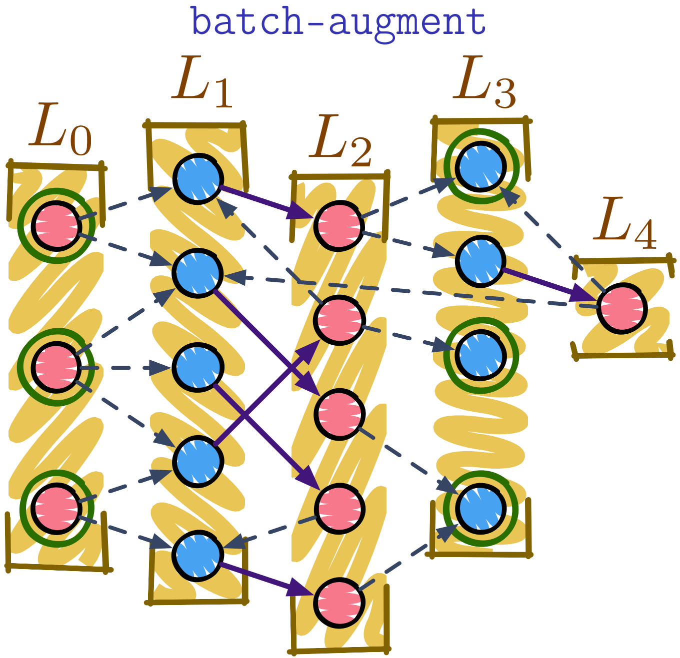
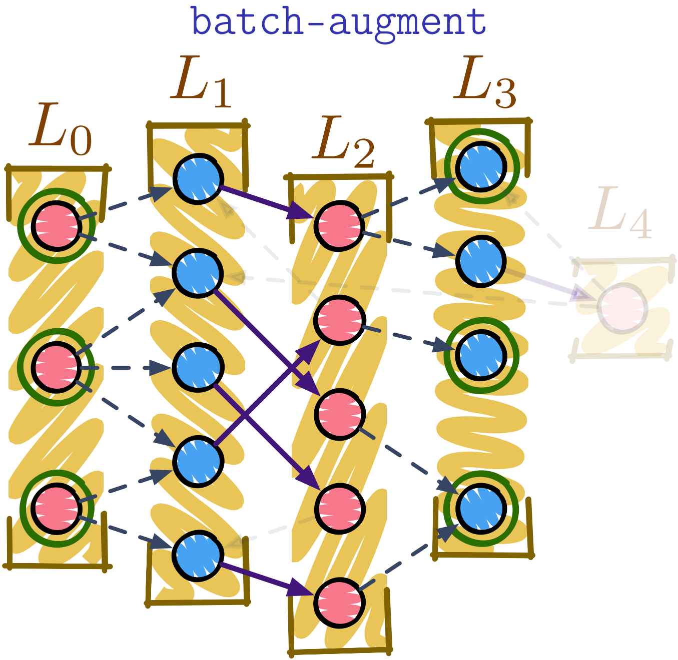



batch-augment
L_0
L_1
L_2
L_3
L_4



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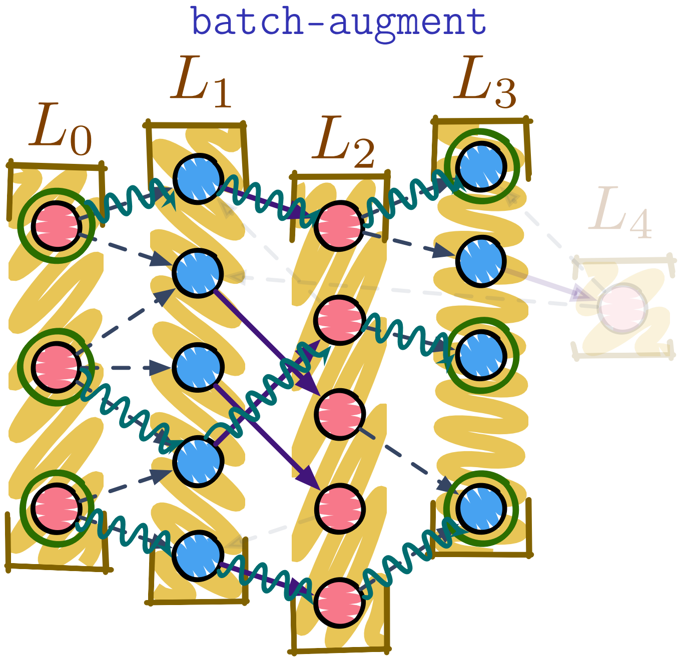

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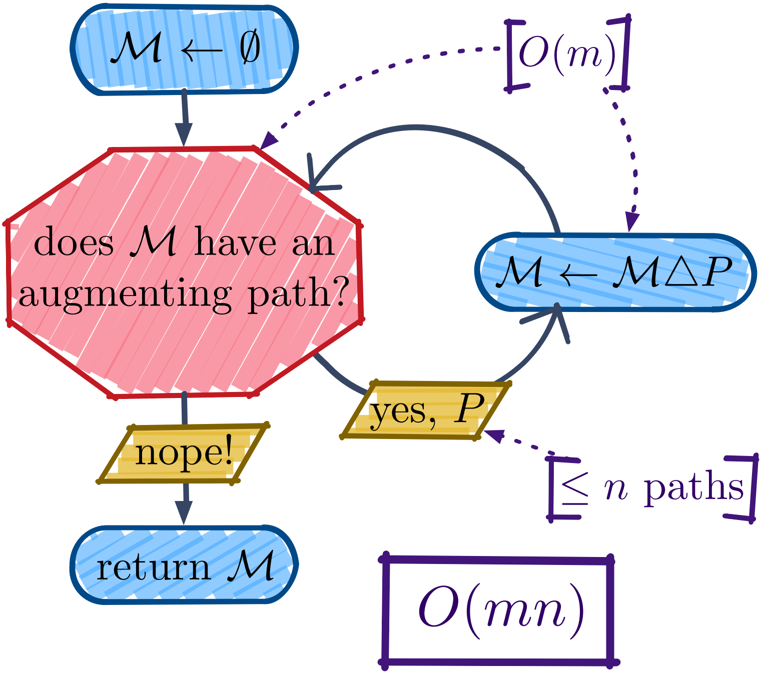
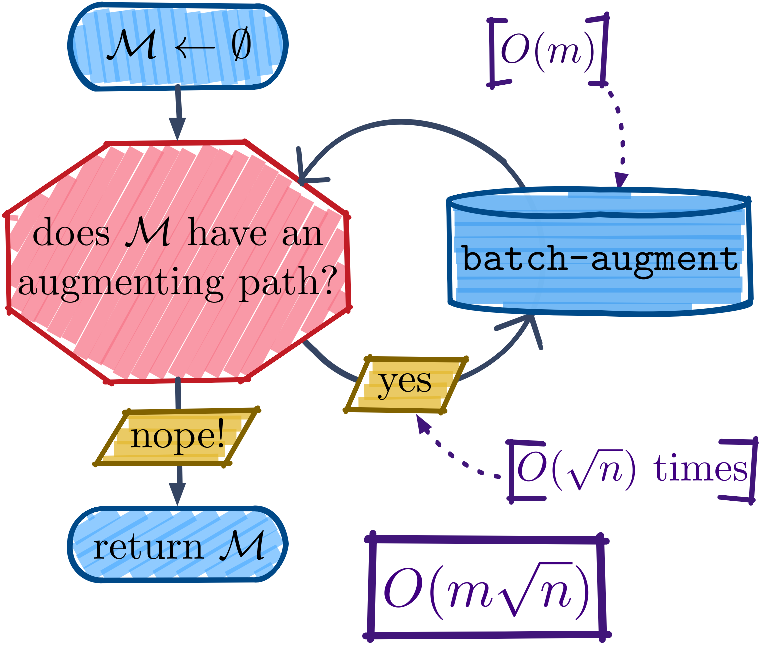
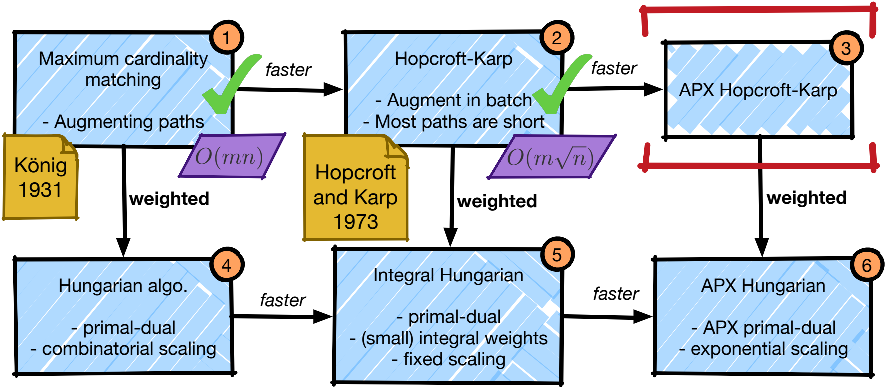
L_3
L_4



$\mathcal{M} \leftarrow \emptyset$
$[O(m)]$
does $\mathcal{M}$ have an augmenting path?
$\mathcal{M} \leftarrow \mathcal{M} \triangle P$
nope!
yes, P
$[\leq n$ paths$]$
return $\mathcal{M}$
$O(mn)$



$\mathcal{M} \leftarrow \emptyset$
does $\mathcal{M}$ have an augmenting path?
yes
batch-augment
$[O(m)]$
$[O(\sqrt{n})$ times$]$
nope!
return $\mathcal{M}$
$O(m\sqrt{n})$

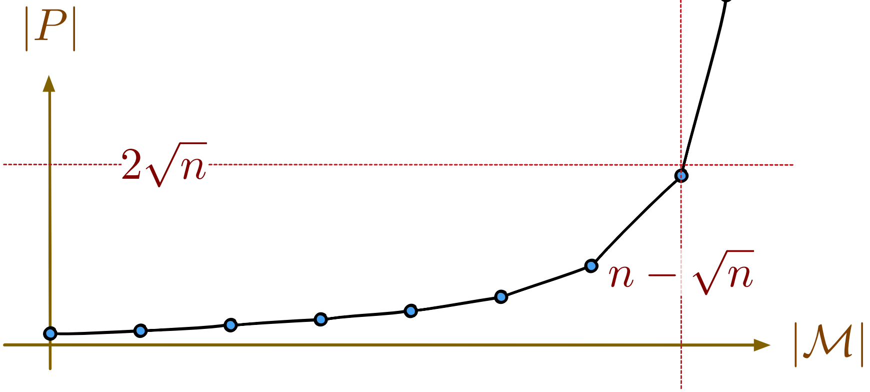


1
Maximum cardinality matching
- Augmenting paths
König 1931
$O(mn)$
faster
2
Hopcroft-Karp
- Augment in batch
- Most paths are short
Hopcroft and Karp 1973
$O(m\sqrt{n})$
faster
3
APX Hopcroft-Karp
weighted
weighted
weighted
4
Hungarian algo.
- primal-dual
- combinatorial scaling
faster
5
Integral Hungarian
- primal-dual
- (small) integral weights
- fixed scaling
faster
6
APX Hungarian
- APX primal-dual
- exponential scaling

Matching $\mathcal{M}$, $k \stackrel{\text{def}}{=} |\mathcal{M}| < \text{OPT}$

Lemma. *$\mathcal{M}$ has aug. path P of length*

$$|P| \le 2\left\lfloor \frac{k}{\text{OPT} - k} \right\rfloor + 1$$



Matching $\mathcal{M}$, $k \stackrel{\text{def}}{=} |\mathcal{M}| < \text{OPT}$

Lemma. *$\mathcal{M}$ has aug. path P of length*

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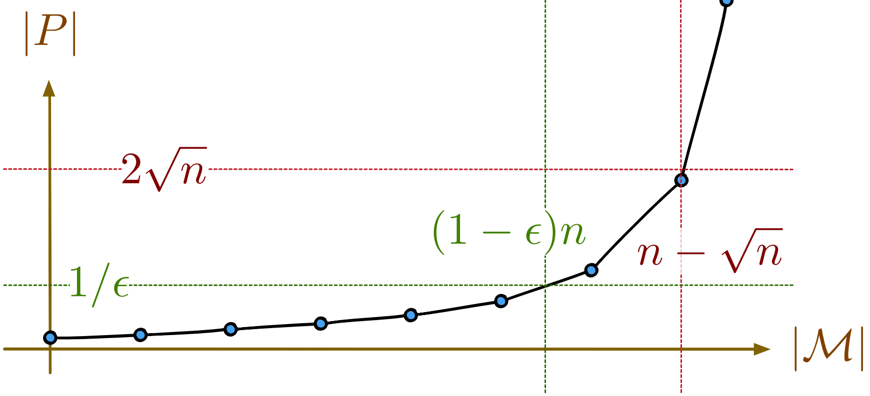
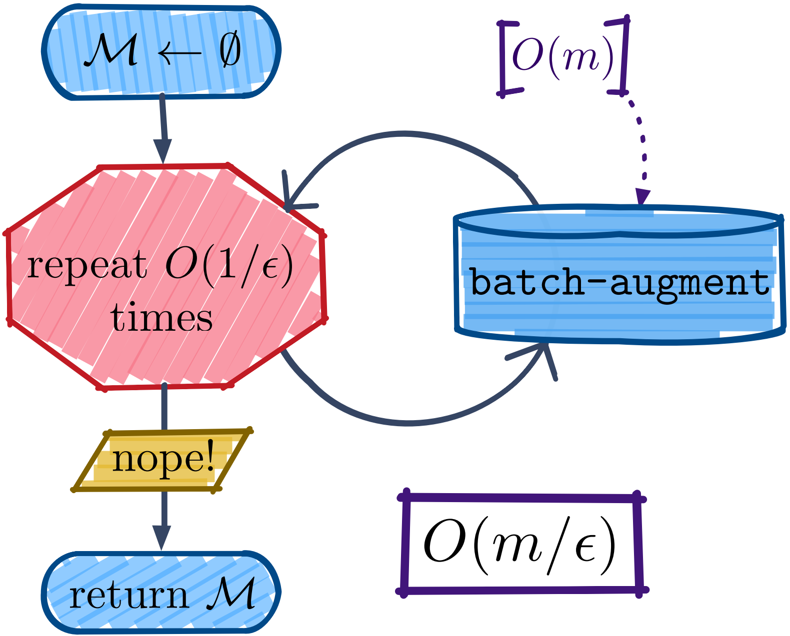
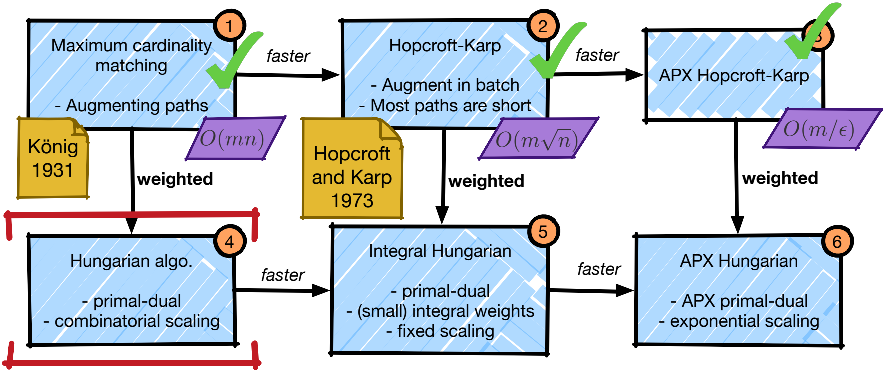



$\mathcal{M} \leftarrow \emptyset$
repeat $O(1/\epsilon)$ times
nope!
return $\mathcal{M}$
batch-augment
$O(m)$
$O(m/\epsilon)$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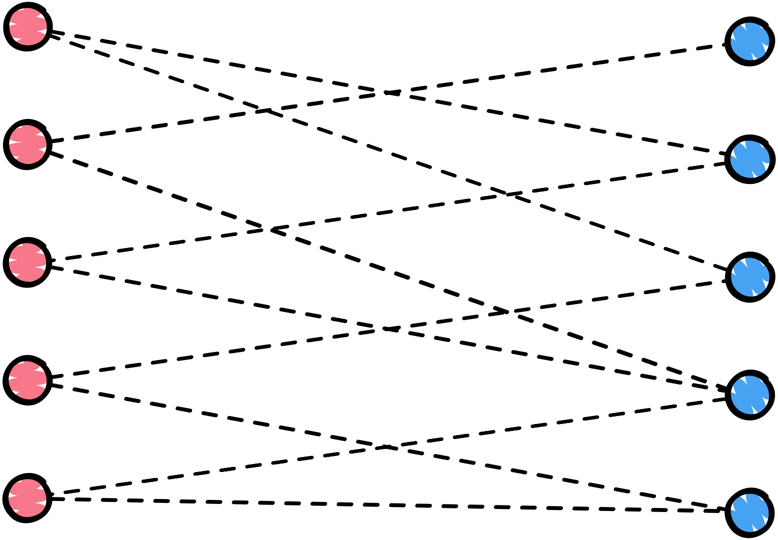
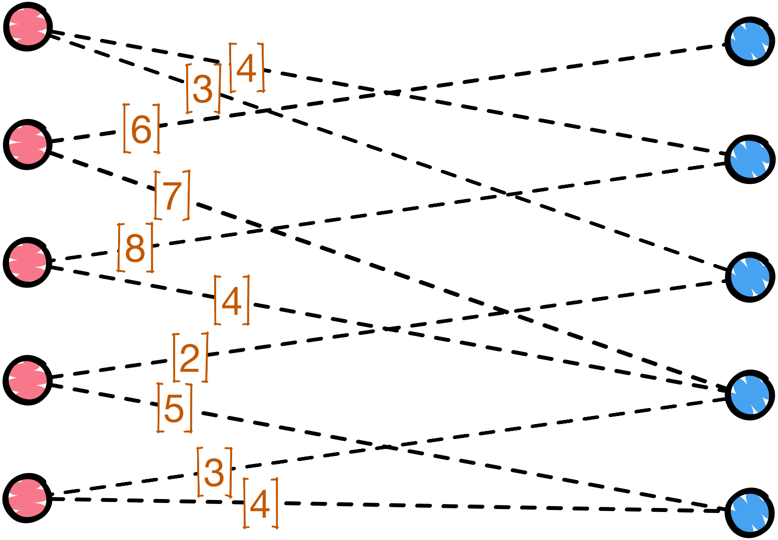
Weighted matchings



Weighted matchings

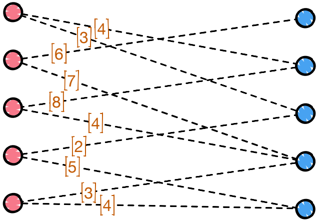


Primal

$$\max_x \quad \sum_{e \in E} w(e) \cdot x(e)$$

$$\text{s.t.} \quad \sum_{e \in \delta(v)} x(e) \leq 1 \qquad v \in L \cup R,$$

$$x \geq 0 \qquad e \in E$$



Primal

$$\max_{x} \quad \sum_{e \in E} w(e) \cdot x(e)$$

$$\text{s.t.} \quad \sum_{e \in \delta(v)} x(e) \leq 1 \qquad v \in L \cup R,$$

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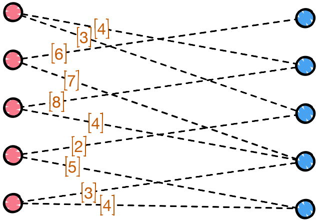


Dual

$$\min_{y} \quad \sum_{v \in L \cup R} y(v)$$

$$\text{s.t.} \quad y(\ell) + y(r) \geq w(\{\ell, r\}) \qquad \{\ell, r\} \in E$$

$$y(v) \geq 0 \qquad v \in L \cup R$$

Primal

$$
\begin{aligned}
\max_{x} \quad & \sum_{e \in E} w(e) \cdot x(e) \\
\text{s.t.} \quad & \sum_{e \in \delta(v)} x(e) \leq 1 \quad && \forall v, \\
& x \geq 0 && \forall e
\end{aligned}
$$

x

Dual

$$
\begin{aligned}
\min_{y} \quad & \sum_{v \in L \cup R} y(v) \\
\text{s.t.} \quad & y(\ell) + y(r) \geq w(e) \quad && \forall e = \{\ell, r\} \\
& y(v) \geq 0 && \forall v
\end{aligned}
$$

y

Primal

$$\max_x \sum_{e \in E} w(e) \cdot x(e)$$

$$\text{s.t.} \quad \sum_{e \in \delta(v)} x(e) \leq 1 \quad \forall v,$$

$$x \geq 0 \quad \forall e$$

x

Dual

$$\min_y \sum_{v \in L \cup R} y(v)$$

$$\text{s.t.} \quad y(\ell) + y(r) \geq w(e) \quad \forall e = \{\ell, r\}$$

$$y(v) \geq 0 \quad \forall v$$

y

x, y optimal if:

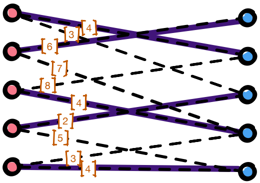
Orthogonality

$$x(\{\ell, r\}) > 0 \Rightarrow y(\ell) + y(r) = w(\{\ell, r\}) \quad \{\ell, r\} \in E$$

$$y(\ell) > 0 \Rightarrow \sum_{e \in \delta(\ell)} x(e) = 1 \quad \ell \in L$$

$$y(r) > 0 \Rightarrow x(e) = 1 \quad r \in R$$

Primal



$\mathcal{M}$

Dual

$$y(\ell) + y(r) \geq w(\{\ell, r\}) \qquad \{\ell, r\} \in E$$

$$y(v) \geq 0 \qquad v \in L \cup R$$

y

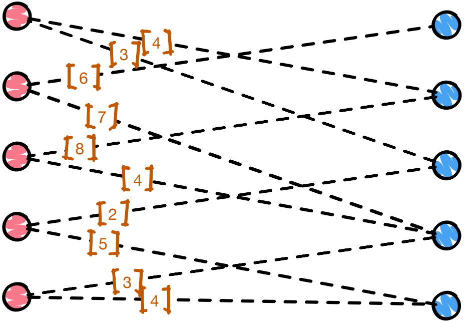
Orthogonality

$$\{\ell, r\} \in \mathcal{M} \Rightarrow y(\ell) + y(r) = w(\{\ell, r\}) \qquad \{\ell, r\} \in E$$

$$y(\ell) > 0 \Rightarrow \ell \in V(\mathcal{M}) \qquad \ell \in L$$

$$y(r) > 0 \Rightarrow r \in V(\mathcal{M}) \qquad r \in R$$

$$w(\mathcal{M}) = \mathrm{OPT}$$



Dual

$$y(\ell) + y(r) \geq w(\{\ell, r\}) \qquad \{\ell, r\} \in E$$

$$y(v) \geq 0 \qquad v \in L \cup R$$

Orthogonality

$$y(\ell) + y(r) = w(\{\ell, r\}) \qquad \forall \{\ell, r\} \in \mathcal{M}$$

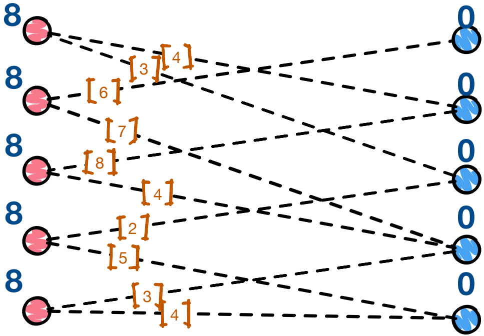
$$\ell \in V(\mathcal{M}) \qquad \forall \ell : y(\ell) > 0$$

$$r \in V(\mathcal{M}) \qquad \forall r : y(r) > 0$$

$$\mathcal{M} \leftarrow \emptyset$$

$$y(\ell) \leftarrow \max_{e \in E} w(e) \quad \forall \ell \in L$$

$$y(r) \leftarrow 0 \quad \forall r \in R$$



Dual

$y(\ell) + y(r) \geq w(\{\ell, r\})$	$\{\ell, r\} \in E$
$y(v) \geq 0$	$v \in L \cup R$

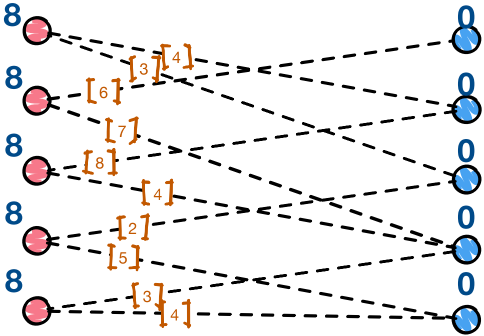
Orthogonality

$y(\ell) + y(r) = w(\{\ell, r\})$	$\forall \{\ell, r\} \in \mathcal{M}$
$\ell \in V(\mathcal{M})$	$\forall \ell : y(\ell) > 0$
$r \in V(\mathcal{M})$	$\forall r : y(r) > 0$

$$\mathcal{M} \leftarrow \emptyset$$

$$y(\ell) \leftarrow \max_{e \in E} w(e) \quad \forall \ell \in L$$

$$y(r) \leftarrow 0 \quad \forall r \in R$$



Dual

$$y(\ell) + y(r) \geq w(\{\ell, r\}) \qquad \{\ell, r\} \in E$$

$$y(v) \geq 0 \qquad v \in L \cup R$$

Orthogonality

$$y(\ell) + y(r) = w(\{\ell, r\}) \qquad \forall \{\ell, r\} \in \mathcal{M}$$

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$$r \in V(\mathcal{M}) \qquad \forall r : y(r) > 0$$

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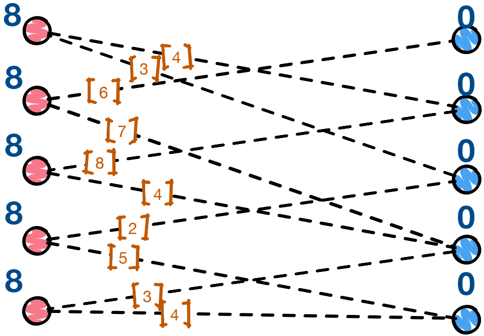
$$y(\ell) \leftarrow \max_{e \in E} w(e) \qquad \forall \ell \in L$$

$$y(r) \leftarrow 0 \quad \forall r \in R$$



Dual

$$y(\ell) + y(r) \geq w(\{\ell, r\}) \qquad \{\ell, r\} \in E$$

$$y(v) \geq 0 \qquad v \in L \cup R$$

GOAL

push $y(\ell) \downarrow 0$

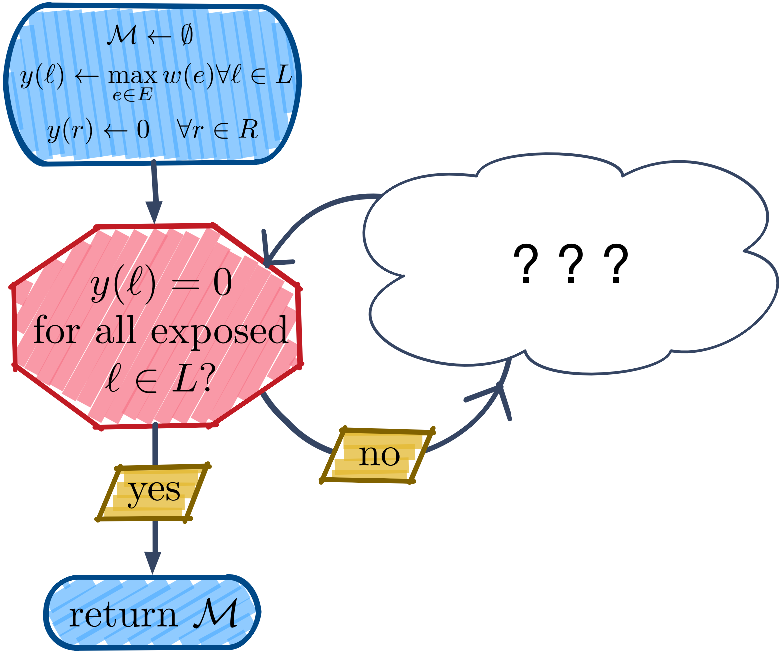
$\forall$ exposed $\ell \in L$

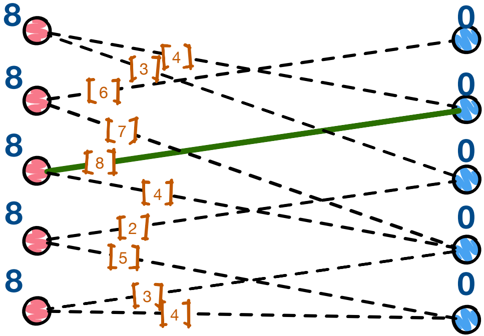
Orthogonality

$$y(\ell) + y(r) = w(\{\ell, r\}) \qquad \forall \{\ell, r\} \in \mathcal{M}$$

$$\ell \in V(\mathcal{M}) \qquad \forall \ell : y(\ell) > 0$$

$$r \in V(\mathcal{M}) \qquad \forall r : y(r) > 0$$


$\mathcal{M} \leftarrow \emptyset$
$y(\ell) \leftarrow \max_{e \in E} w(e) \forall \ell \in L$
$y(r) \leftarrow 0 \quad \forall r \in R$
$y(\ell) = 0$ for all exposed $\ell \in L$?
? ? ?
no
yes
return $\mathcal{M}$



Dual

$$y(\ell) + y(r) \geq w(\{\ell, r\}) \qquad \{\ell, r\} \in E$$

$$y(v) \geq 0 \qquad v \in L \cup R$$

Orthogonality

$$y(\ell) + y(r) = w(\{\ell, r\}) \qquad \forall \{\ell, r\} \in \mathcal{M}$$

$$\ell \in V(\mathcal{M}) \qquad \forall \ell : y(\ell) > 0$$

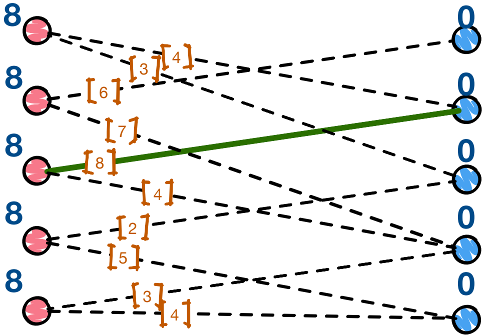
$$r \in V(\mathcal{M}) \qquad \forall r : y(r) > 0$$

GOAL

push $y(\ell) \downarrow 0$

$\forall$ exposed $\ell \in L$

we can add
“tight” edges
to our matching



Dual

$$y(\ell) + y(r) \geq w(\{\ell, r\}) \qquad \{\ell, r\} \in E$$

$$y(v) \geq 0 \qquad v \in L \cup R$$

GOAL

push $y(\ell) \downarrow 0$

$\forall$ exposed $\ell \in L$

Orthogonality

$$y(\ell) + y(r) = w(\{\ell, r\}) \qquad \forall \{\ell, r\} \in \mathcal{M}$$

$$\ell \in V(\mathcal{M}) \qquad \forall \ell : y(\ell) > 0$$

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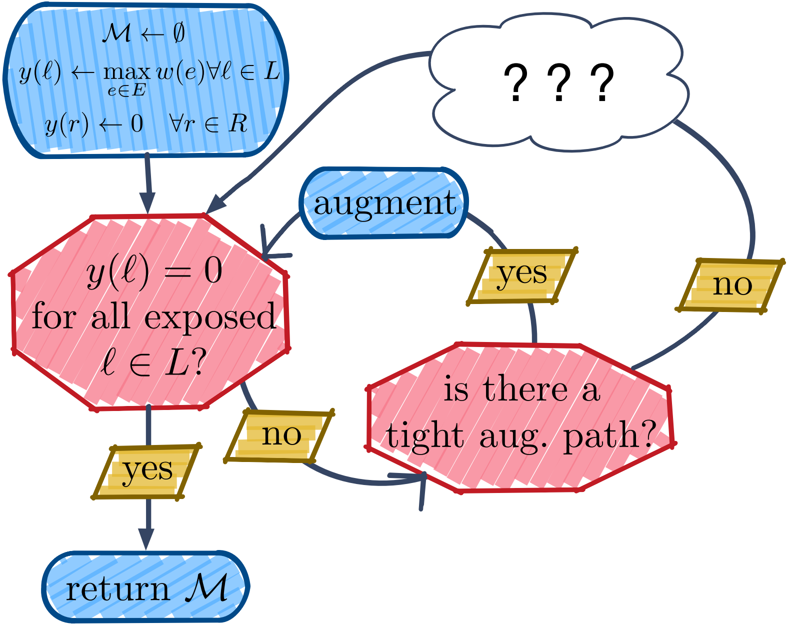
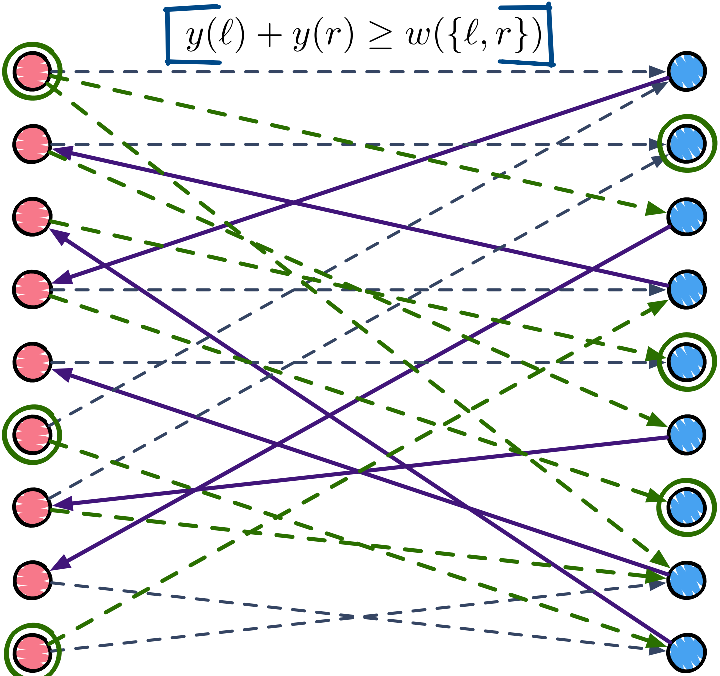
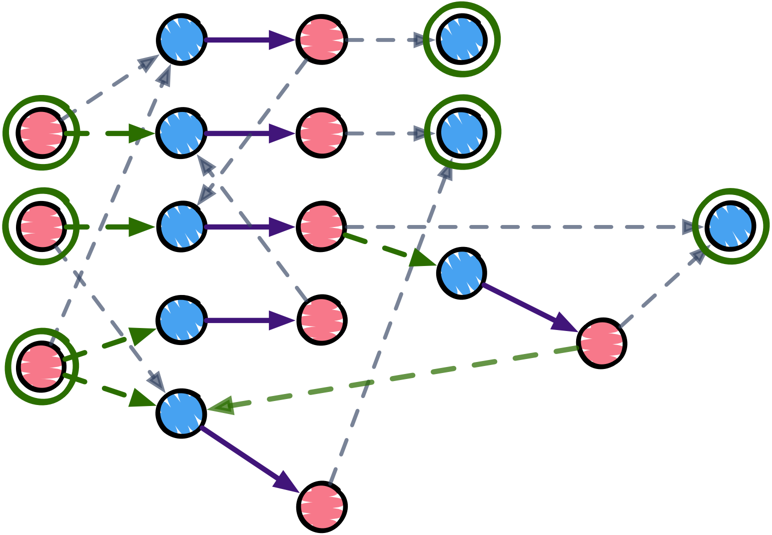


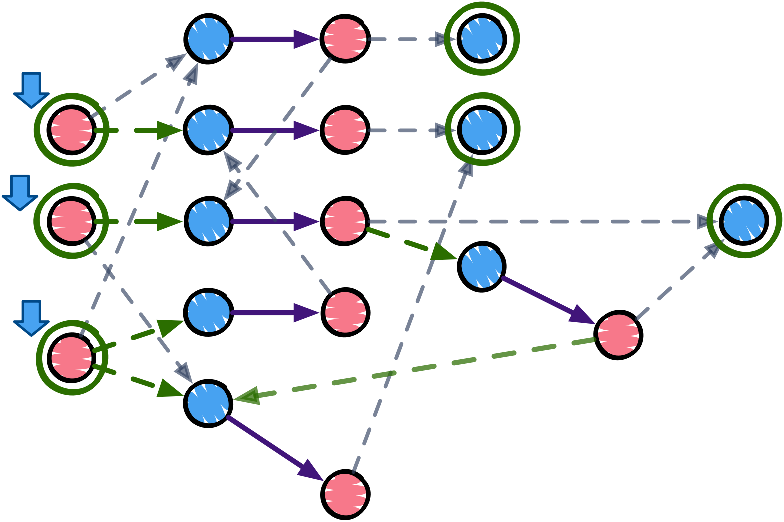
$\mathcal{M} \leftarrow \emptyset$
$y(\ell) \leftarrow \max_{e \in E} w(e) \forall \ell \in L$
$y(r) \leftarrow 0 \quad \forall r \in R$
? ? ?
augment
$y(\ell) = 0$ for all exposed $\ell \in L$?
yes
no
yes
no
is there a tight aug. path?
return $\mathcal{M}$


$y(\ell) + y(r) \geq w(\{\ell, r\})$

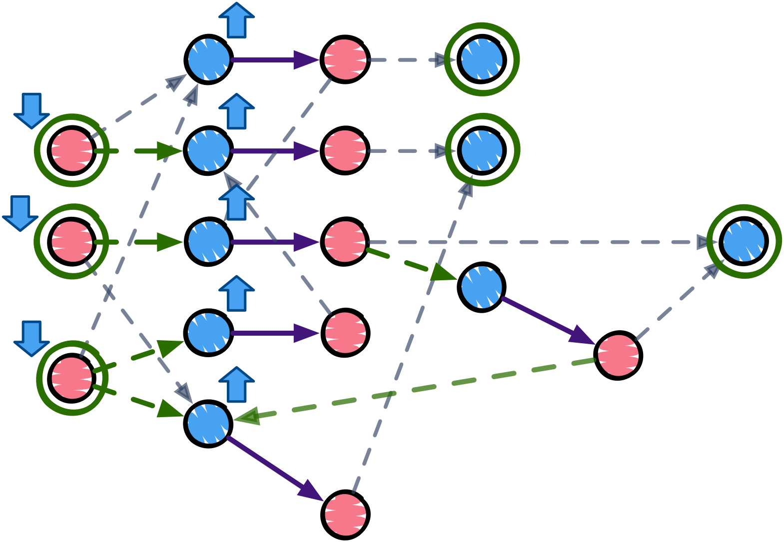
$$y(\ell) + y(r) \geq w(\{\ell, r\})$$



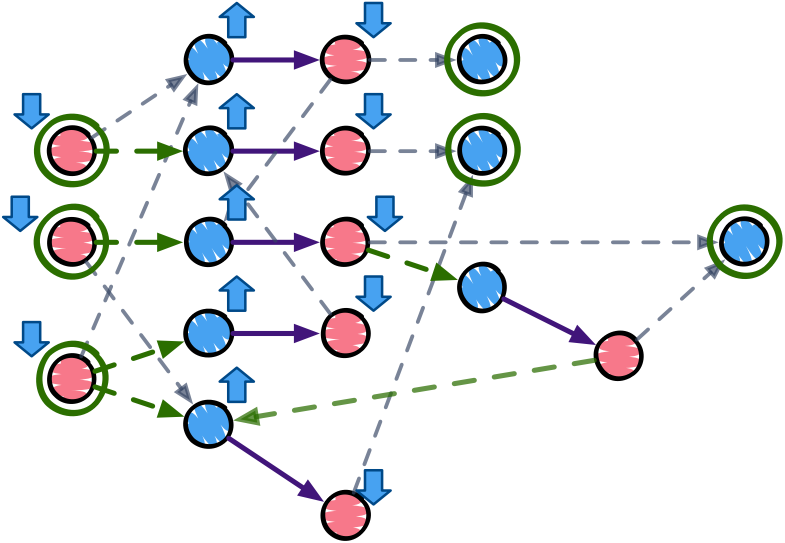
$$y(\ell) + y(r) \geq w(\{\ell, r\})$$



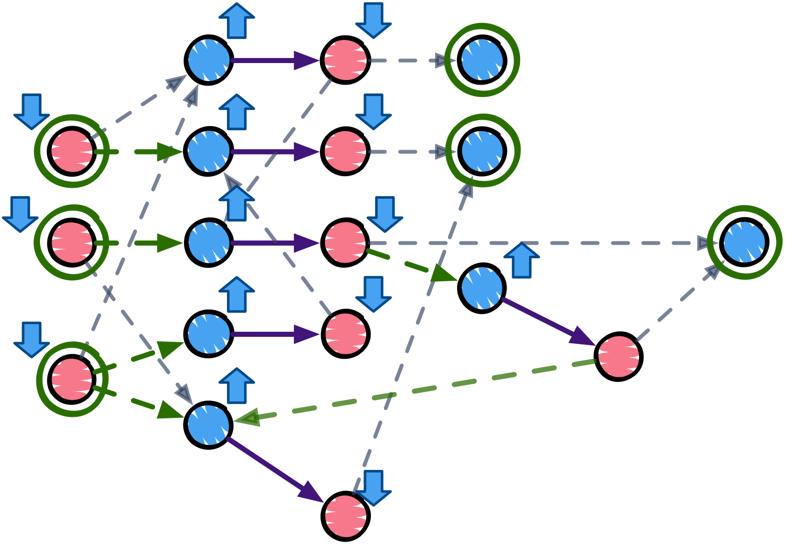
$$y(\ell) + y(r) \geq w(\{\ell, r\})$$



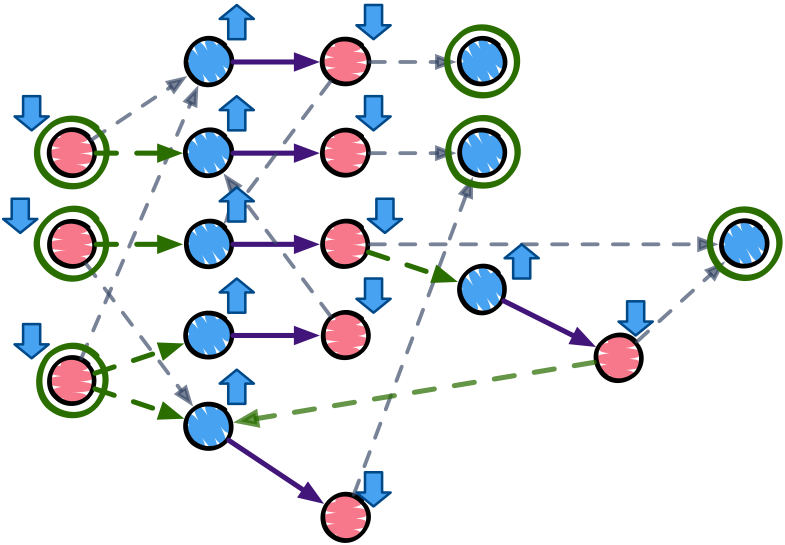
$$y(\ell) + y(r) \geq w(\{\ell, r\})$$

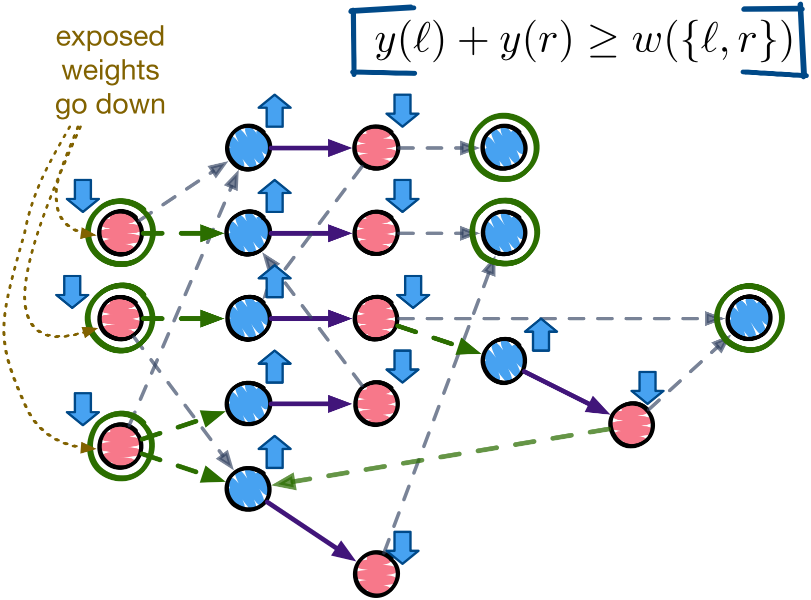


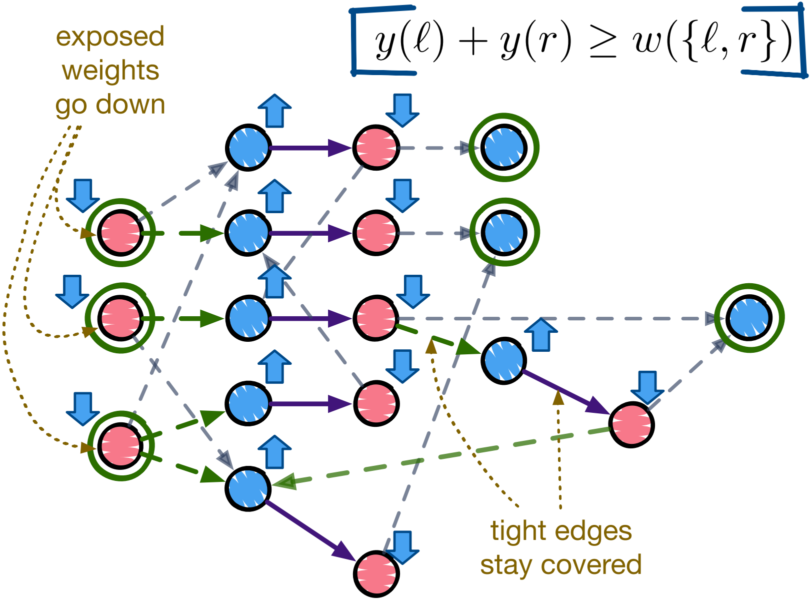
$$y(\ell) + y(r) \geq w(\{\ell, r\})$$

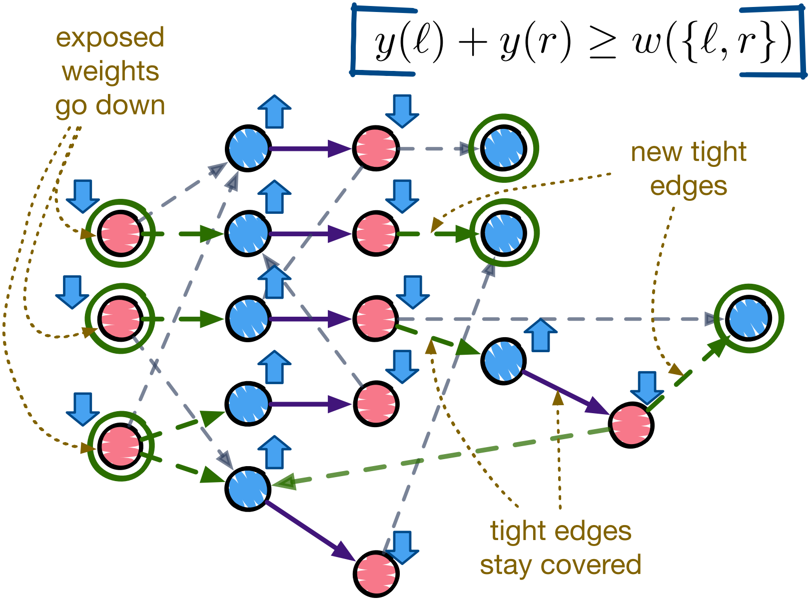


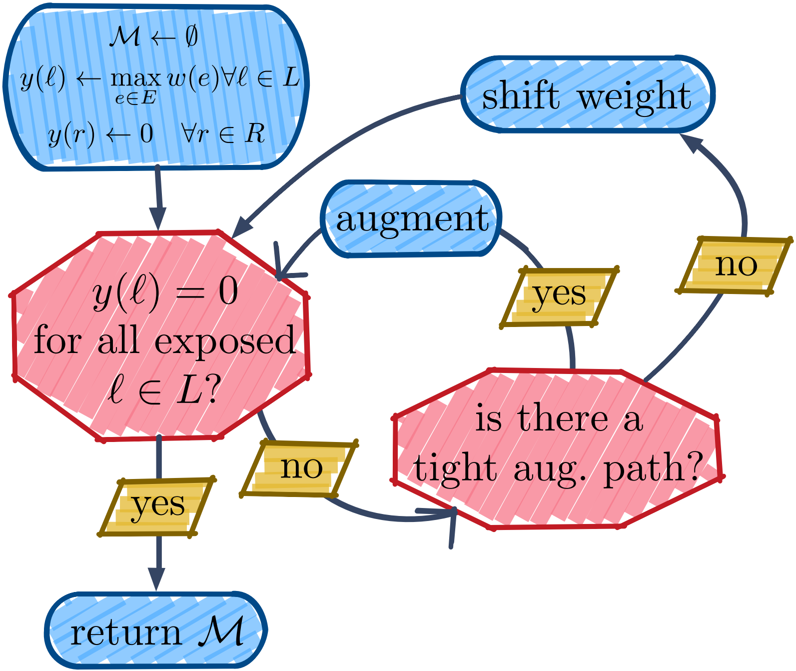
$$y(\ell) + y(r) \geq w(\{\ell, r\})$$



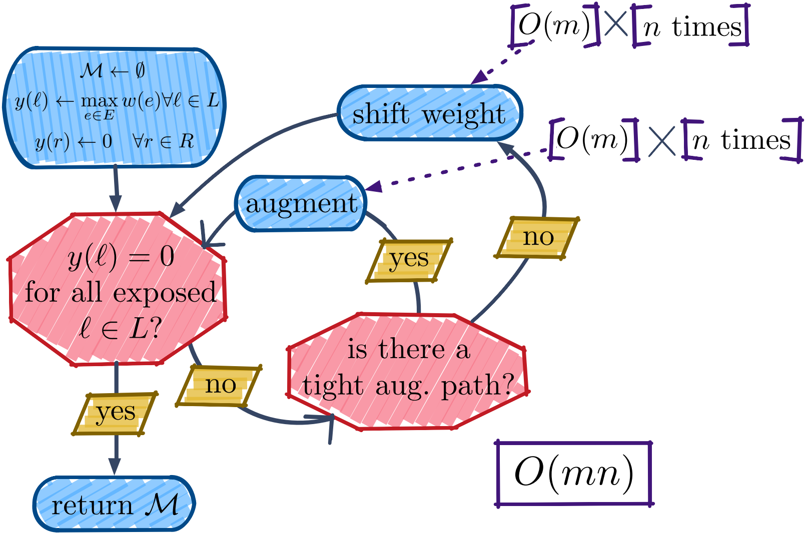

exposed
weights
go down
$y(\ell) + y(r) \geq w(\{\ell, r\})$


exposed weights go down
$y(\ell) + y(r) \geq w(\{\ell, r\})$
tight edges stay covered


$y(\ell) + y(r) \geq w(\{\ell, r\})$
exposed weights go down
new tight edges
tight edges stay covered



$\mathcal{M} \leftarrow \emptyset$
$y(\ell) \leftarrow \max_{e \in E} w(e) \forall \ell \in L$
$y(r) \leftarrow 0 \quad \forall r \in R$
shift weight
augment
$y(\ell) = 0$ for all exposed $\ell \in L$?
yes
no
no
is there a tight aug. path?
yes
return $\mathcal{M}$



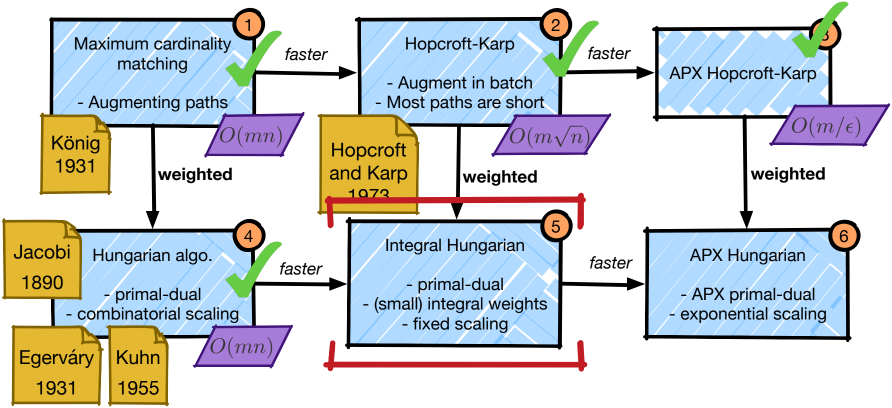
$\mathcal{M} \leftarrow \emptyset$
$y(\ell) \leftarrow \max_{e \in E} w(e) \forall \ell \in L$
$y(r) \leftarrow 0 \quad \forall r \in R$
shift weight
augment
$O(m)$ × n times
$O(m)$ × n times
$y(\ell) = 0$ for all exposed $\ell \in L$?
yes
no
is there a tight aug. path?
yes
no
return $\mathcal{M}$
$O(mn)$

Using Egerváry's reduction and Konig's maximum matching alorithm, in the fall of 1953 I solved several 12 by 12 assignment problems (with 3-digit integers as data) by hand. Each of these examples took under two hours to solve and I was convinced that the combined algorithm was 'good'.

This must have been one of the last times when pencil and paper could beat the largest and fastest electronic computer in the world.

- Harold Kuhn,

On the origin of the Hungarian method



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Hungarian algo.
- primal-dual
- combinatorial scaling
$O(mn)$
Jacobi 1890
Egerváry 1931
Kuhn 1955
faster
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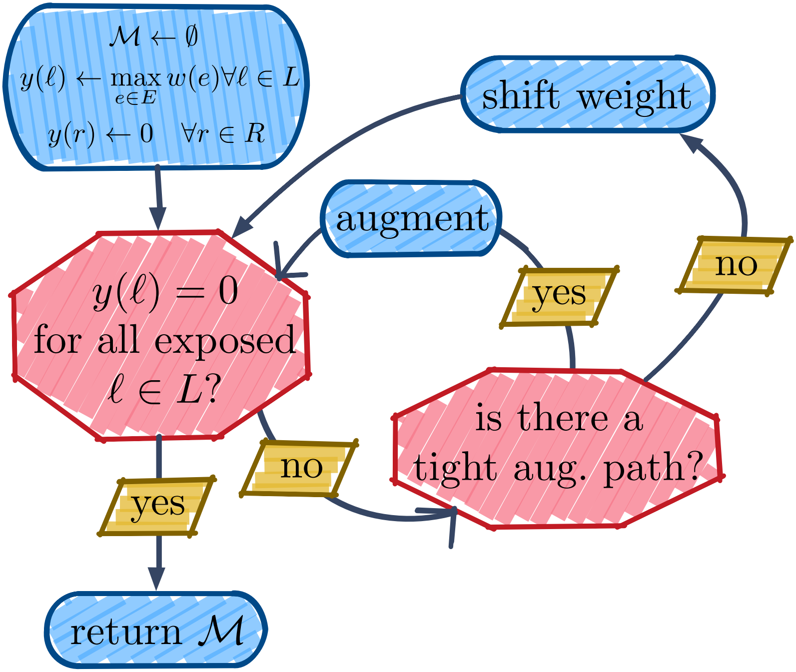
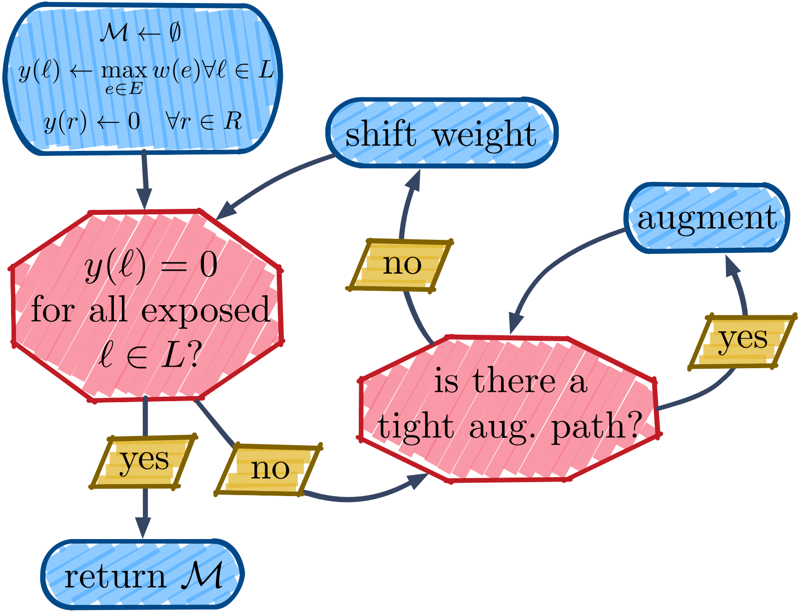
Integral Hungarian
- primal-dual
- (small) integral weights
- fixed scaling
faster
6
APX Hungarian
- APX primal-dual
- exponential scaling

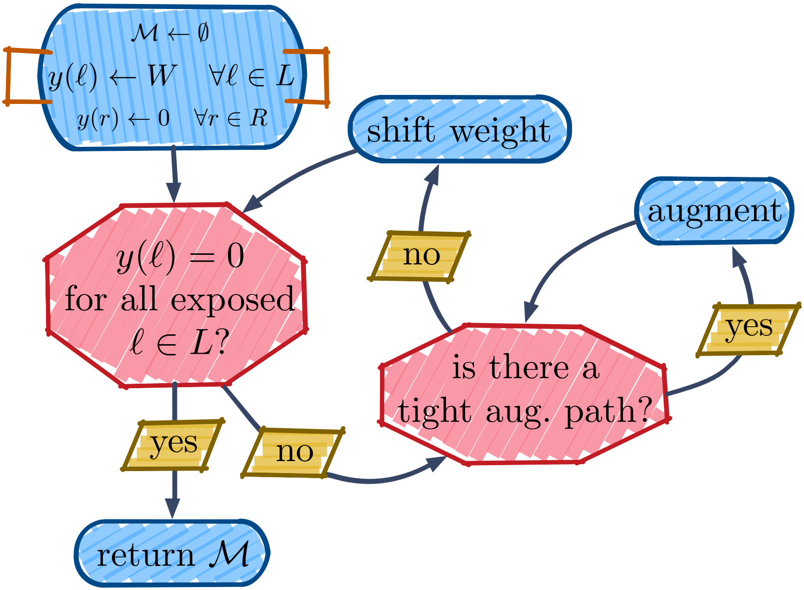
What if the weights are small and integral?

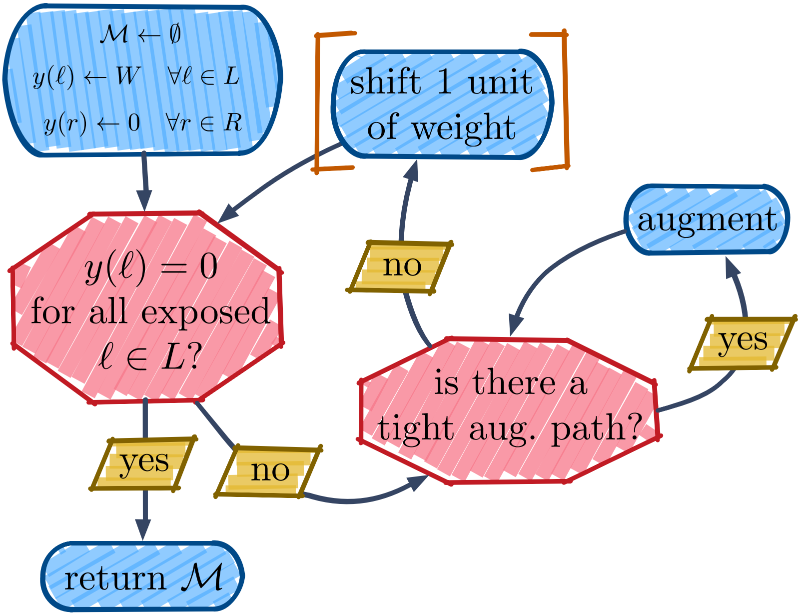
$$w(e) \in \{1, \ldots, W\} \qquad \forall e \in E$$

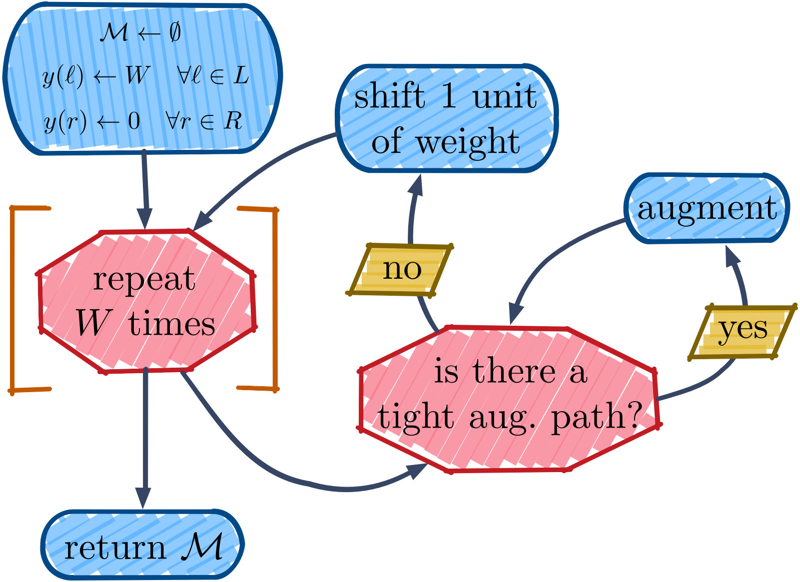

$\mathcal{M} \leftarrow \emptyset$
$y(\ell) \leftarrow \max_{e \in E} w(e) \forall \ell \in L$
$y(r) \leftarrow 0 \quad \forall r \in R$
shift weight
augment
$y(\ell) = 0$ for all exposed $\ell \in L$?
yes
no
yes
no
is there a tight aug. path?
return $\mathcal{M}$

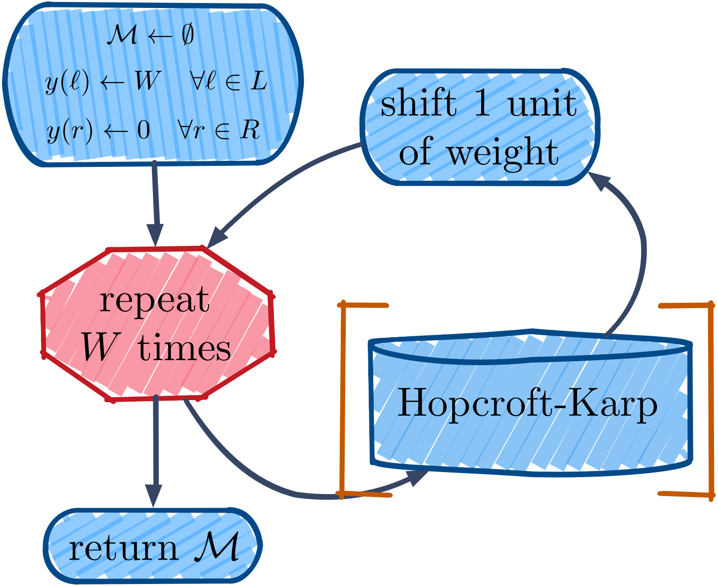


$\mathcal{M} \leftarrow \emptyset$
$y(\ell) \leftarrow \max_{e \in E} w(e) \forall \ell \in L$
$y(r) \leftarrow 0 \quad \forall r \in R$
shift weight
augment
$y(\ell) = 0$ for all exposed $\ell \in L$?
no
yes
is there a tight aug. path?
yes
no
return $\mathcal{M}$

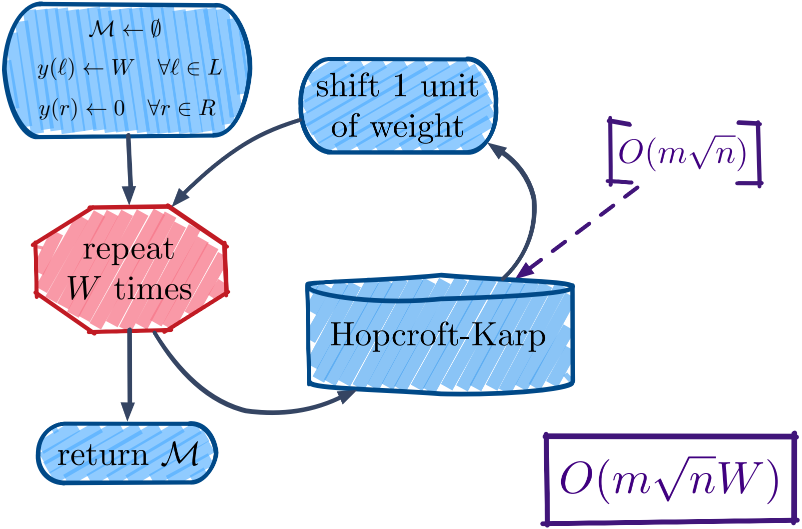

$\mathcal{M} \leftarrow \emptyset$
$y(\ell) \leftarrow W \quad \forall \ell \in L$
$y(r) \leftarrow 0 \quad \forall r \in R$
shift weight
augment
$y(\ell) = 0$ for all exposed $\ell \in L$?
no
yes
is there a tight aug. path?
yes
no
return $\mathcal{M}$

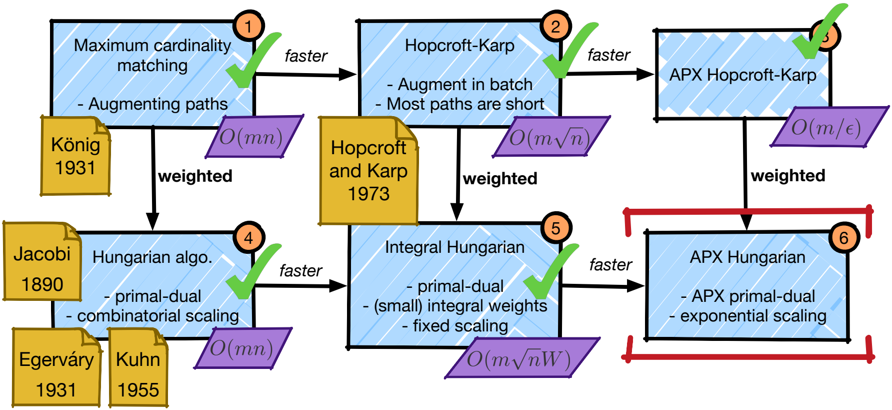

$\mathcal{M} \leftarrow \emptyset$
$y(\ell) \leftarrow W \quad \forall \ell \in L$
$y(r) \leftarrow 0 \quad \forall r \in R$
shift 1 unit of weight
$y(\ell) = 0$ for all exposed $\ell \in L$?
yes
no
return $\mathcal{M}$
is there a tight aug. path?
no
yes
augment


$\mathcal{M} \leftarrow \emptyset$
$y(\ell) \leftarrow W \quad \forall \ell \in L$
$y(r) \leftarrow 0 \quad \forall r \in R$
shift 1 unit of weight
augment
repeat W times
no
yes
is there a tight aug. path?
return $\mathcal{M}$



$\mathcal{M} \leftarrow \emptyset$
$y(\ell) \leftarrow W \quad \forall \ell \in L$
$y(r) \leftarrow 0 \quad \forall r \in R$
shift 1 unit of weight
repeat W times
Hopcroft-Karp
return $\mathcal{M}$


$\mathcal{M} \leftarrow \emptyset$
$y(\ell) \leftarrow W \quad \forall \ell \in L$
$y(r) \leftarrow 0 \quad \forall r \in R$
shift 1 unit of weight
$O(m\sqrt{n})$
repeat W times
Hopcroft-Karp
return $\mathcal{M}$
$O(m\sqrt{n}W)$



Maximum cardinality matching
- Augmenting paths
1
$O(mn)$
König 1931
faster
Hopcroft-Karp
- Augment in batch
- Most paths are short
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$O(m\sqrt{n})$
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- combinatorial scaling
4
$O(mn)$
Egerváry 1931
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faster
Integral Hungarian
- primal-dual
- (small) integral weights
- fixed scaling
5
$O(m\sqrt{n}W)$
faster
APX Hungarian
- APX primal-dual
- exponential scaling
6

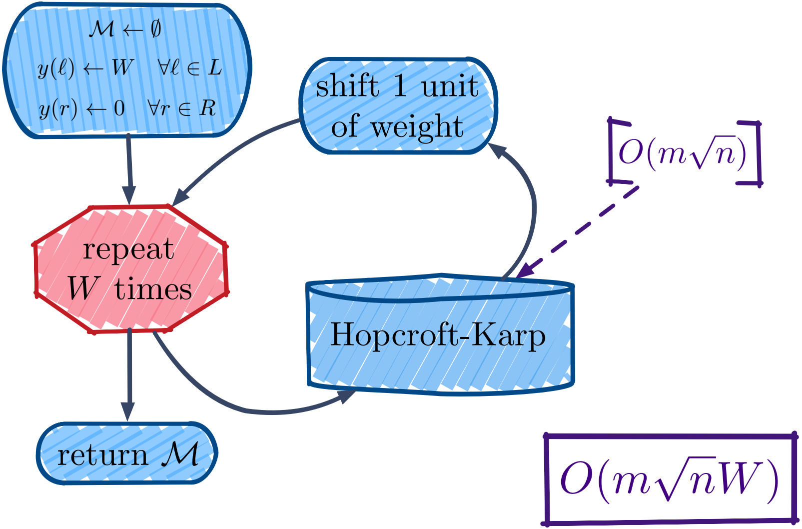
Goal: beat $O(m\sqrt{n})$

1. Approximate primal-dual conditions

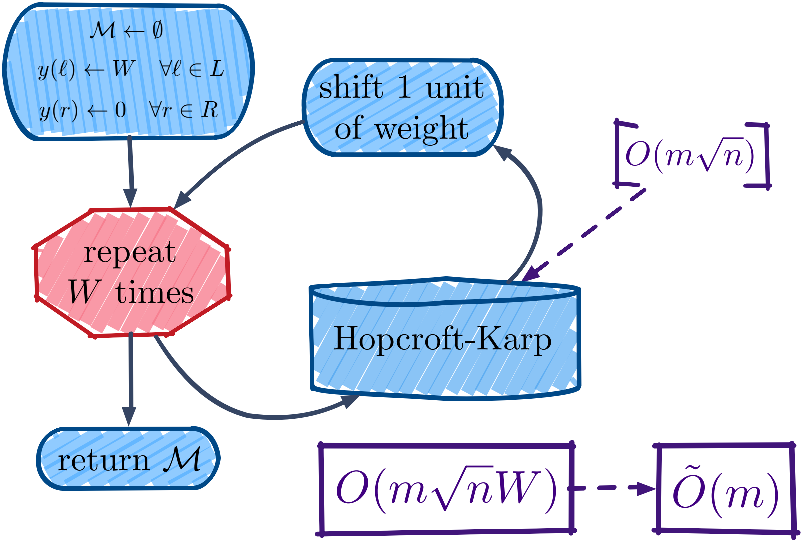
2. Exponential scaling

3. Approximate cardinality at each scale

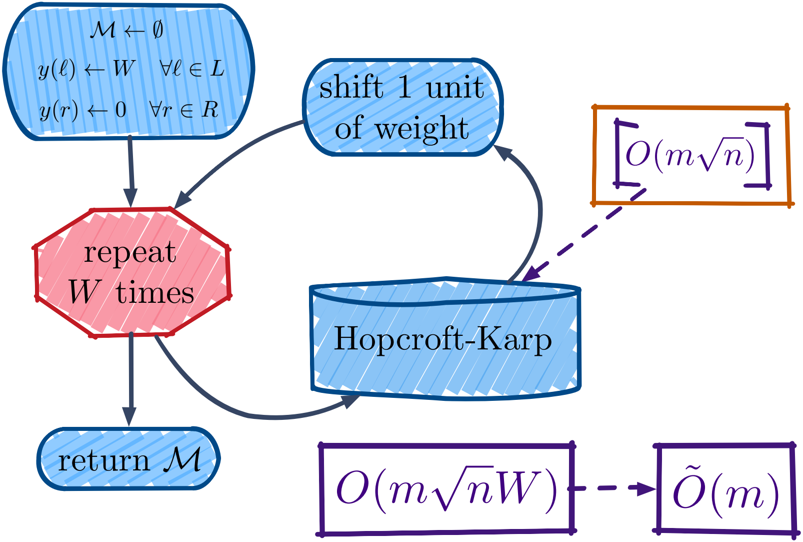
4. Lossy weight shift



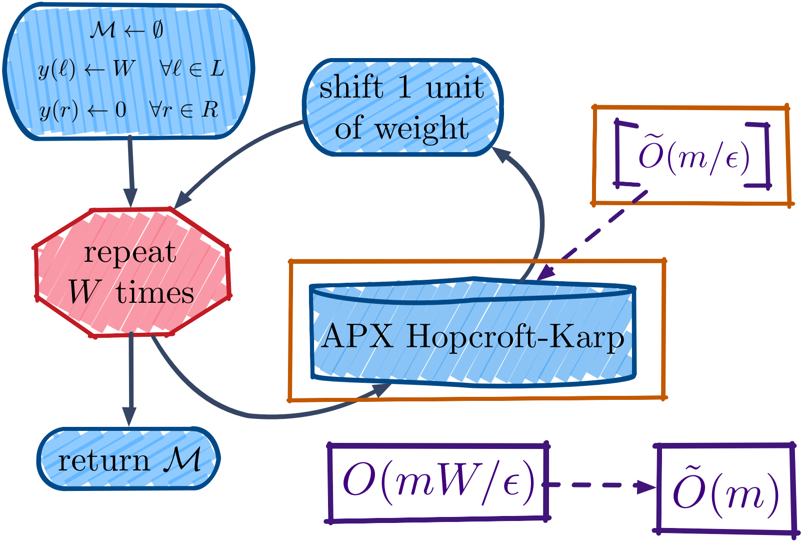
$\mathcal{M} \leftarrow \emptyset$
$y(\ell) \leftarrow W \quad \forall \ell \in L$
$y(r) \leftarrow 0 \quad \forall r \in R$
shift 1 unit of weight
$O(m\sqrt{n})$
repeat W times
Hopcroft-Karp
return $\mathcal{M}$
$O(m\sqrt{n}W)$



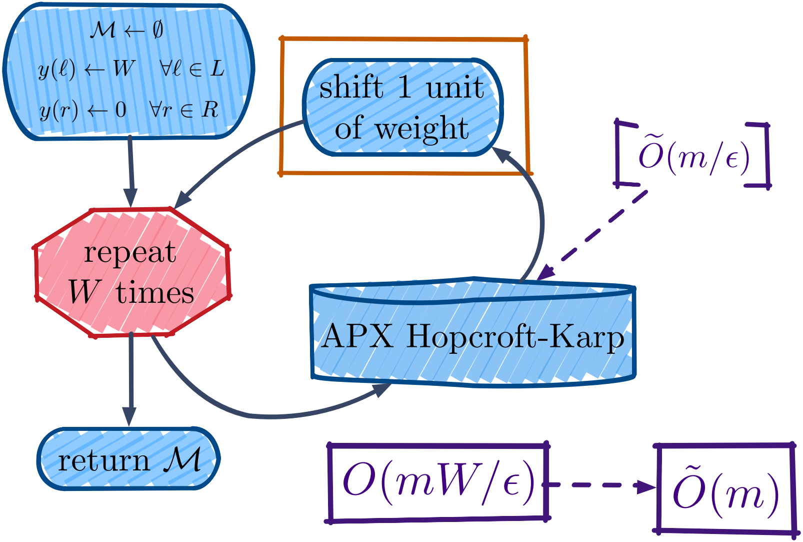
$\mathcal{M} \leftarrow \emptyset$
$y(\ell) \leftarrow W \quad \forall \ell \in L$
$y(r) \leftarrow 0 \quad \forall r \in R$
shift 1 unit of weight
$O(m\sqrt{n})$
repeat W times
Hopcroft-Karp
return $\mathcal{M}$
$O(m\sqrt{n}W)$
$\tilde{O}(m)$



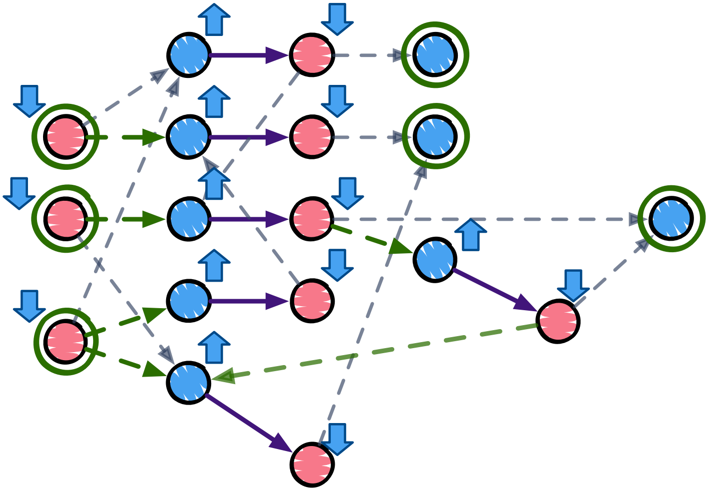
$\mathcal{M} \leftarrow \emptyset$
$y(\ell) \leftarrow W \quad \forall \ell \in L$
$y(r) \leftarrow 0 \quad \forall r \in R$
shift 1 unit of weight
$\left[O(m\sqrt{n})\right]$
repeat W times
Hopcroft-Karp
return $\mathcal{M}$
$O(m\sqrt{n}W)$
$\tilde{O}(m)$



$\mathcal{M} \leftarrow \emptyset$
$y(\ell) \leftarrow W \quad \forall \ell \in L$
$y(r) \leftarrow 0 \quad \forall r \in R$
shift 1 unit of weight
$\left[\tilde{O}(m/\epsilon)\right]$
repeat W times
APX Hopcroft-Karp
return $\mathcal{M}$
$O(mW/\epsilon)$
$\tilde{O}(m)$

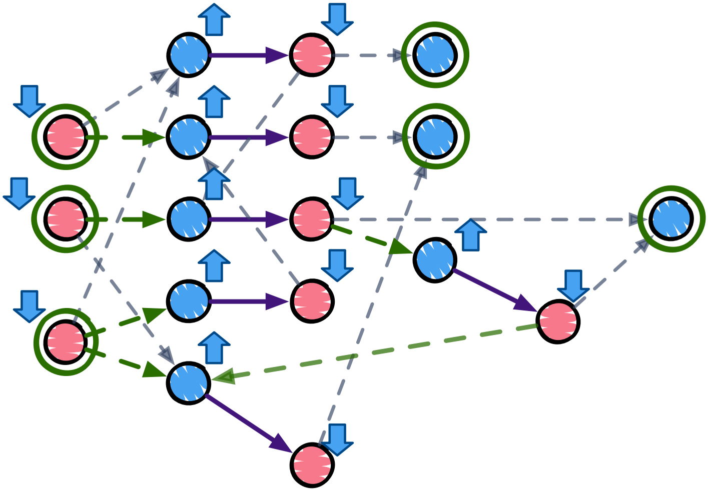

$\mathcal{M} \leftarrow \emptyset$
$y(\ell) \leftarrow W \quad \forall \ell \in L$
$y(r) \leftarrow 0 \quad \forall r \in R$
shift 1 unit of weight
$\widetilde{O}(m/\epsilon)$
repeat W times
APX Hopcroft-Karp
return $\mathcal{M}$
$O(mW/\epsilon)$
$\tilde{O}(m)$

$$y(\ell) + y(r) \geq w(\{\ell, r\})$$



$$y(\ell) + y(r) \geq w(\{\ell, r\})$$

$$y(r) = 0 \quad \forall r \in R \setminus V(\mathcal{M})$$



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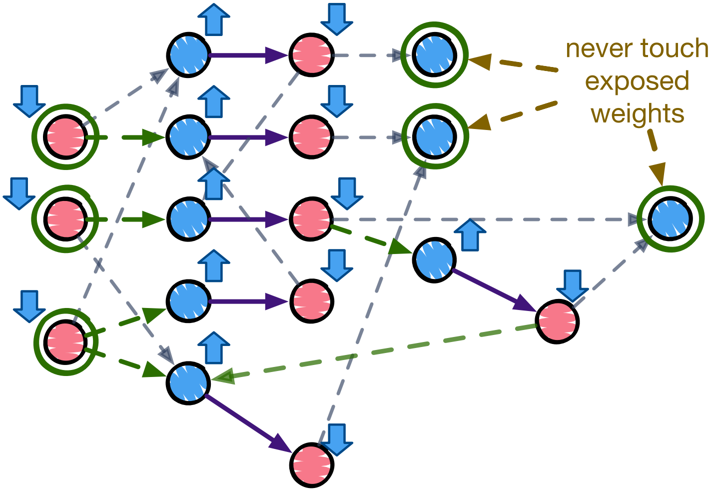
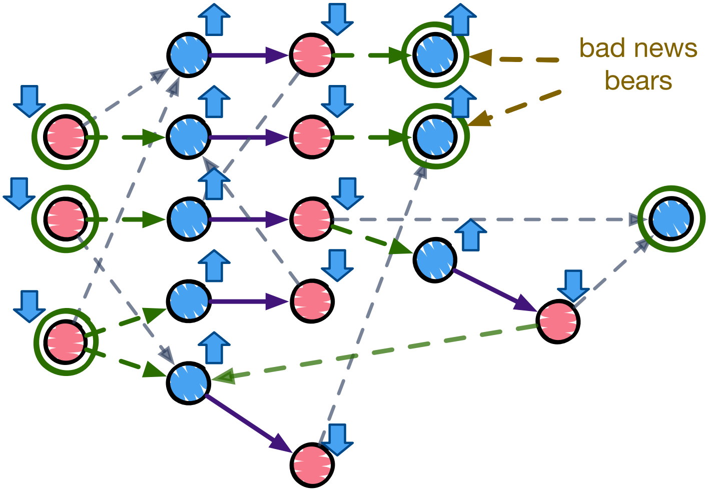
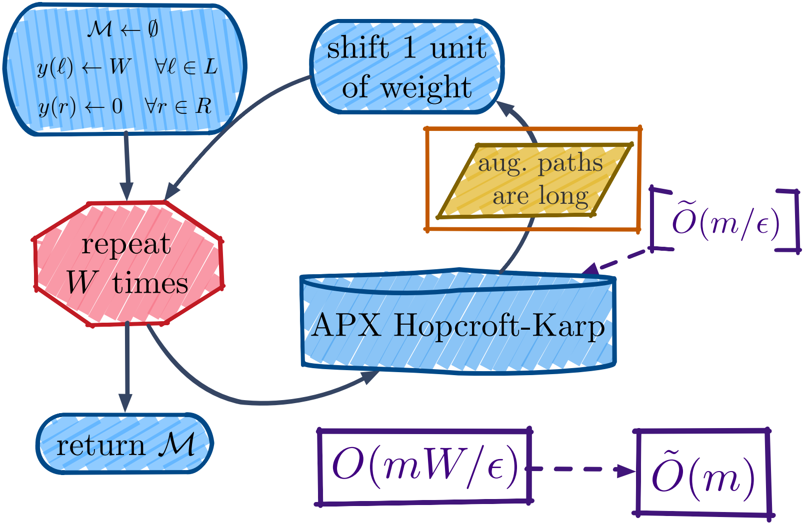


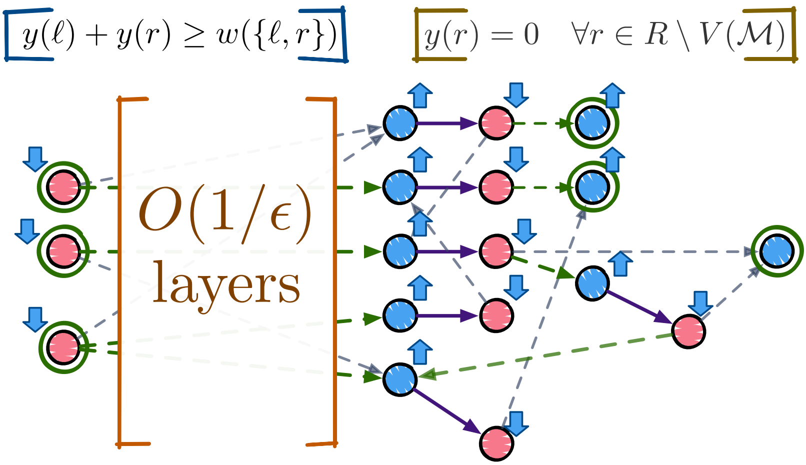
$$y(\ell) + y(r) \geq w(\{\ell, r\})$$

$$y(r) = 0 \quad \forall r \in R \setminus V(\mathcal{M})$$

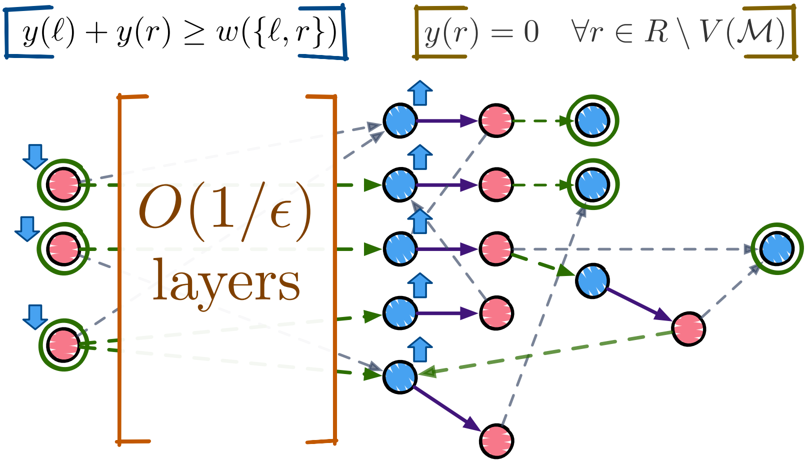




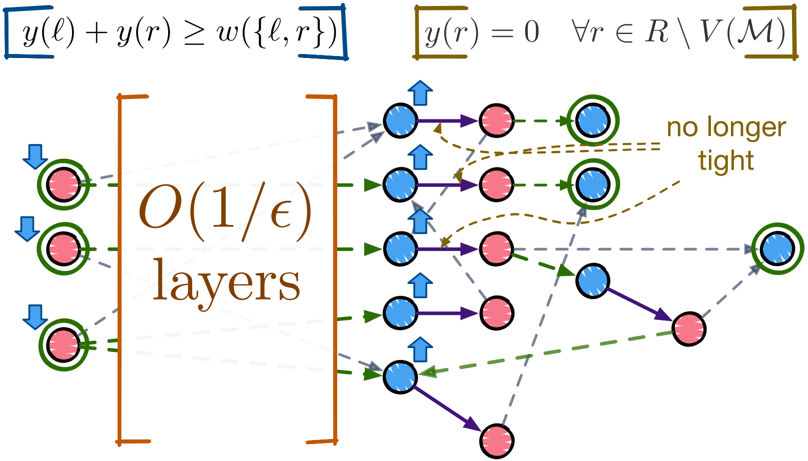
$\mathcal{M} \leftarrow \emptyset$
$y(\ell) \leftarrow W \quad \forall \ell \in L$
$y(r) \leftarrow 0 \quad \forall r \in R$
shift 1 unit of weight
aug. paths are long
$\widetilde{O}(m/\epsilon)$
repeat W times
APX Hopcroft-Karp
return $\mathcal{M}$
$O(mW/\epsilon)$
$\tilde{O}(m)$



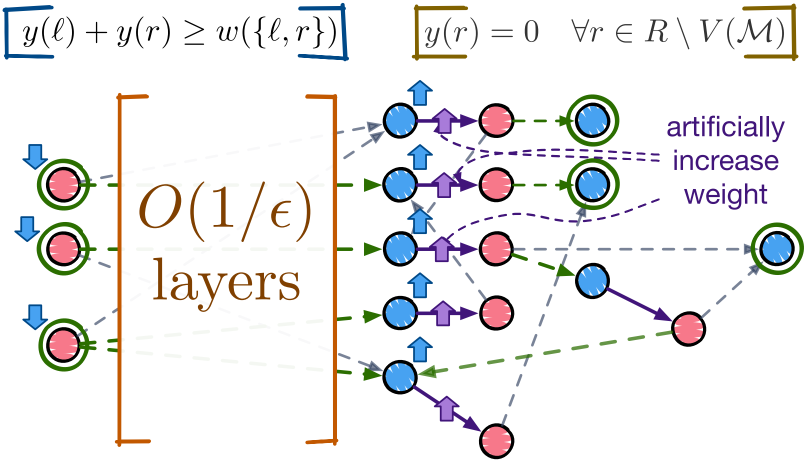
$y(\ell) + y(r) \geq w(\{\ell, r\})$
$y(r) = 0 \quad \forall r \in R \setminus V(\mathcal{M})$
$O(1/\epsilon)$
layers



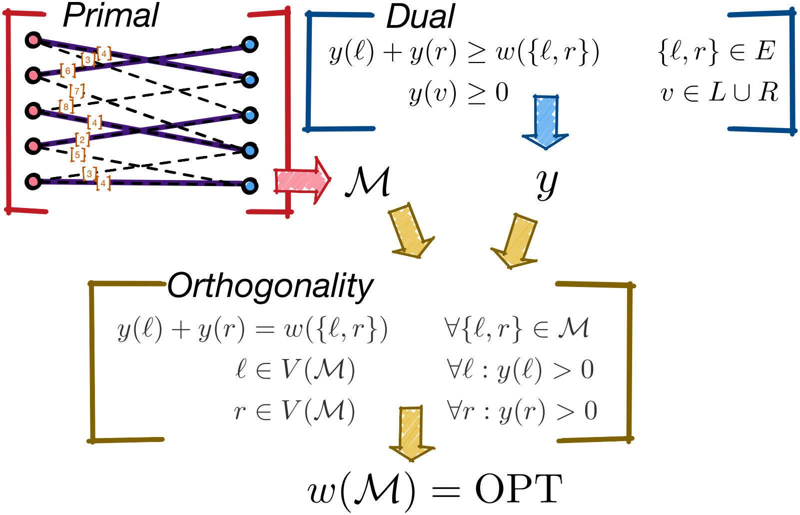
$y(\ell) + y(r) \geq w(\{\ell, r\})$
$y(r) = 0 \quad \forall r \in R \setminus V(\mathcal{M})$
$O(1/\epsilon)$
layers



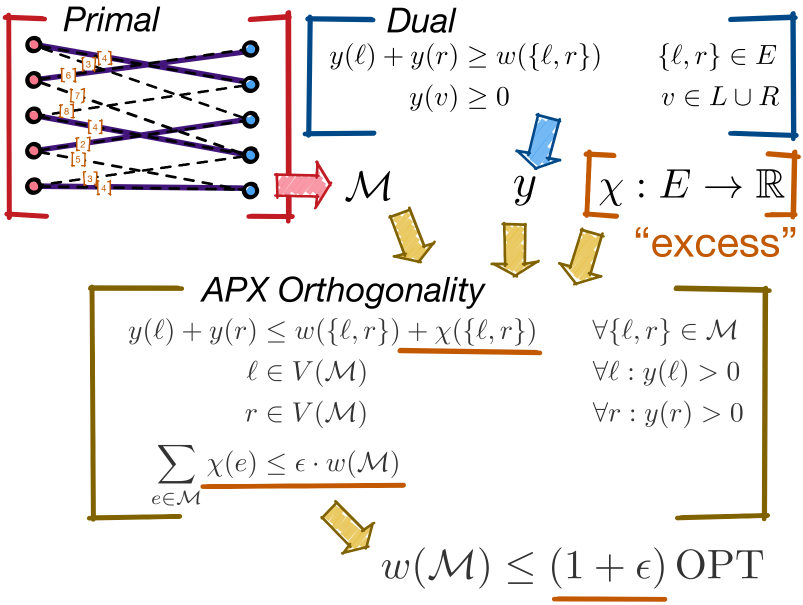
$y(\ell) + y(r) \geq w(\{\ell, r\})$
$y(r) = 0 \quad \forall r \in R \setminus V(\mathcal{M})$
$O(1/\epsilon)$
layers
no longer
tight

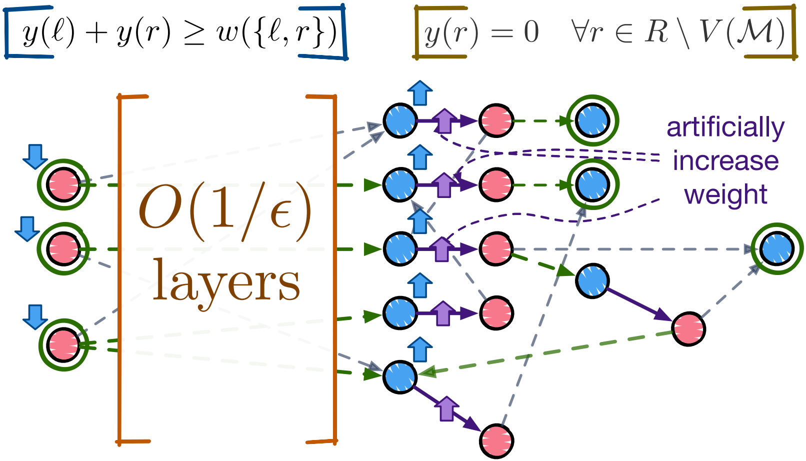


$y(\ell) + y(r) \geq w(\{\ell, r\})$
$y(r) = 0 \quad \forall r \in R \setminus V(\mathcal{M})$
$O(1/\epsilon)$
layers
artificially
increase
weight

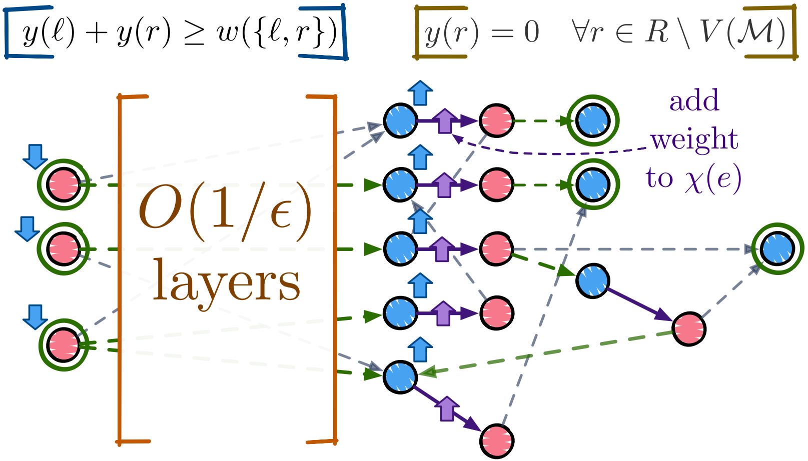


Primal
Dual
$y(\ell) + y(r) \geq w(\{\ell, r\})$ $\{\ell, r\} \in E$
$y(v) \geq 0$ $v \in L \cup R$
$\mathcal{M}$
y
Orthogonality
$y(\ell) + y(r) = w(\{\ell, r\})$ $\forall \{\ell, r\} \in \mathcal{M}$
$\ell \in V(\mathcal{M})$ $\forall \ell : y(\ell) > 0$
$r \in V(\mathcal{M})$ $\forall r : y(r) > 0$
$w(\mathcal{M}) = \mathrm{OPT}$

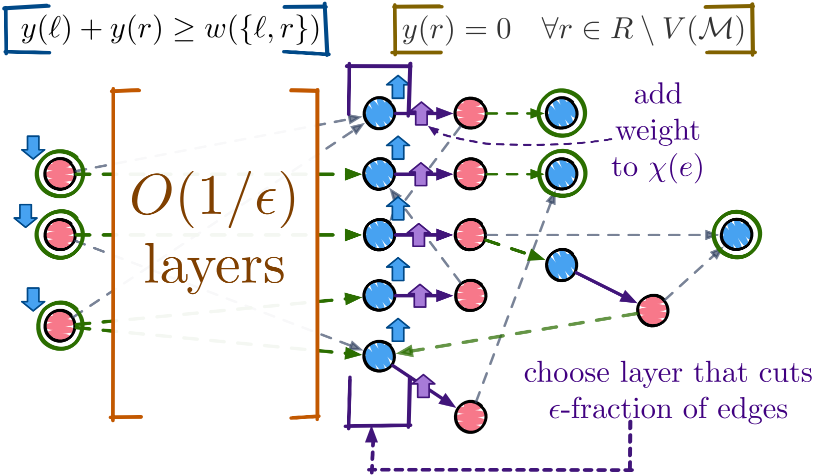

Primal
Dual
$y(\ell) + y(r) \geq w(\{\ell, r\})$ $\{\ell, r\} \in E$
$y(v) \geq 0$ $v \in L \cup R$
$\mathcal{M}$
y
$\chi : E \to \mathbb{R}$
“excess”
APX Orthogonality
$y(\ell) + y(r) \leq w(\{\ell, r\}) + \chi(\{\ell, r\})$ $\forall \{\ell, r\} \in \mathcal{M}$
$\ell \in V(\mathcal{M})$ $\forall \ell : y(\ell) > 0$
$r \in V(\mathcal{M})$ $\forall r : y(r) > 0$
$\sum_{e \in \mathcal{M}} \chi(e) \leq \epsilon \cdot w(\mathcal{M})$
$w(\mathcal{M}) \leq (1 + \epsilon)\,\mathrm{OPT}$



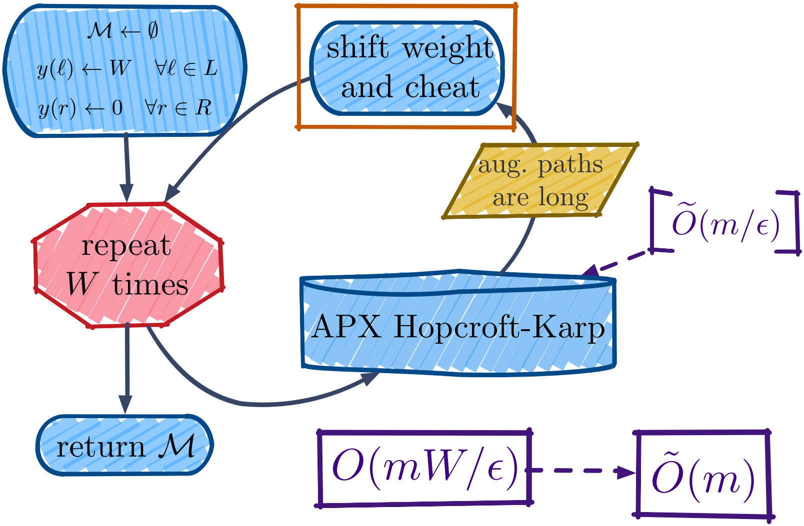
$y(\ell) + y(r) \geq w(\{\ell, r\})$
$y(r) = 0 \quad \forall r \in R \setminus V(\mathcal{M})$
$O(1/\epsilon)$
layers
artificially
increase
weight



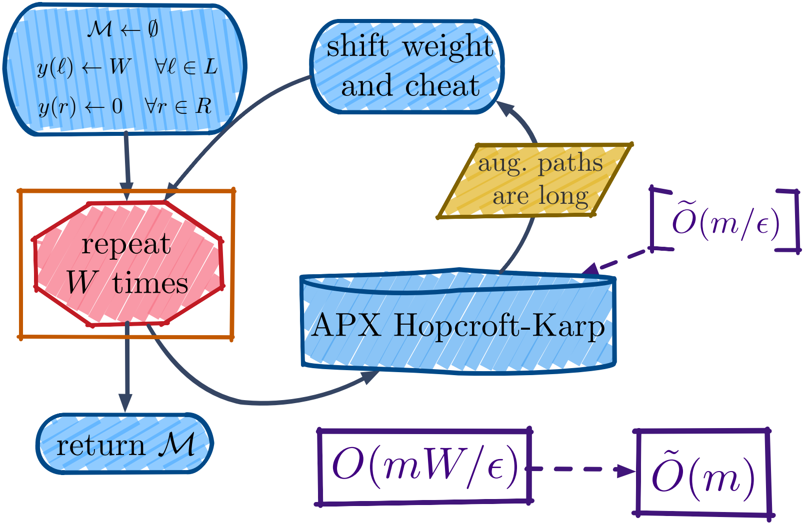
$y(\ell) + y(r) \geq w(\{\ell, r\})$
$y(r) = 0 \quad \forall r \in R \setminus V(\mathcal{M})$
$O(1/\epsilon)$
layers
add
weight
to $\chi(e)$



$y(\ell) + y(r) \geq w(\{\ell, r\})$
$y(r) = 0 \quad \forall r \in R \setminus V(\mathcal{M})$
$O(1/\epsilon)$ layers
add weight to $\chi(e)$
choose layer that cuts ϵ-fraction of edges



$\mathcal{M} \leftarrow \emptyset$
$y(\ell) \leftarrow W \quad \forall \ell \in L$
$y(r) \leftarrow 0 \quad \forall r \in R$
shift weight and cheat
aug. paths are long
$\tilde{O}(m/\epsilon)$
repeat W times
APX Hopcroft-Karp
return $\mathcal{M}$
$O(mW/\epsilon)$
$\tilde{O}(m)$



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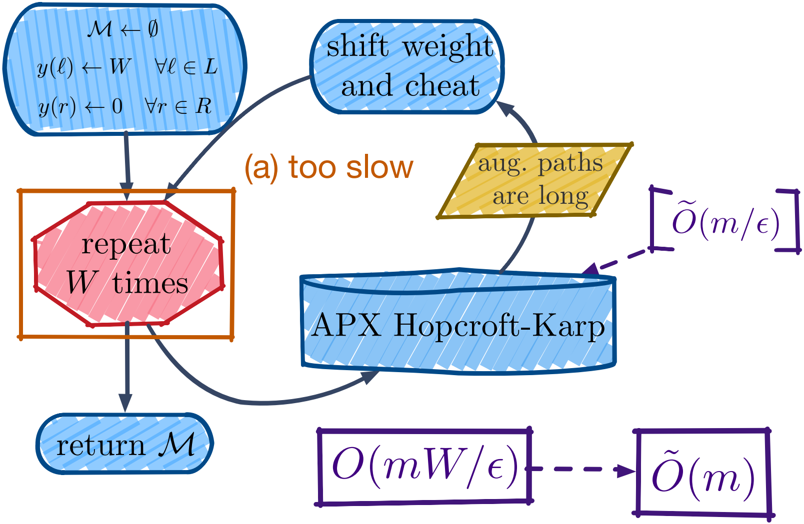
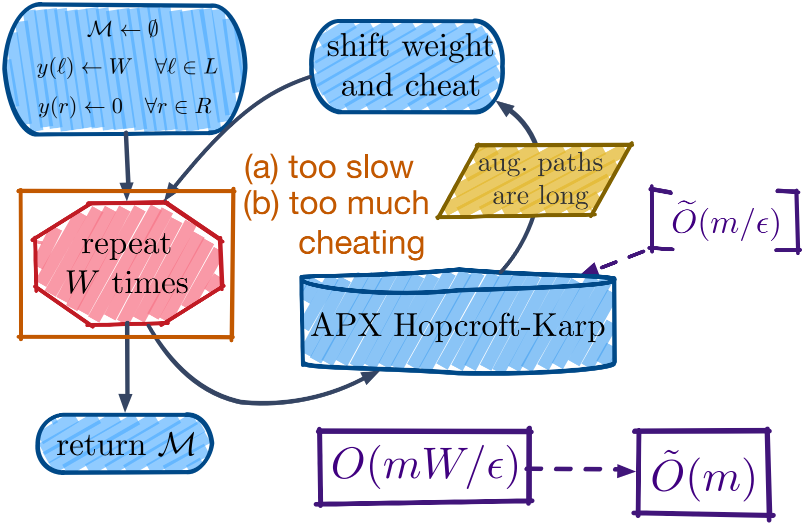
repeat W times
aug. paths are long
$\tilde{O}(m/\epsilon)$
APX Hopcroft-Karp
return $\mathcal{M}$
$O(mW/\epsilon)$
$\tilde{O}(m)$


$\mathcal{M} \leftarrow \emptyset$
$y(\ell) \leftarrow W \quad \forall \ell \in L$
$y(r) \leftarrow 0 \quad \forall r \in R$
shift weight and cheat
(a) too slow
aug. paths are long
$\widetilde{O}(m/\epsilon)$
repeat W times
APX Hopcroft-Karp
return $\mathcal{M}$
$O(mW/\epsilon)$
$\tilde{O}(m)$



$\mathcal{M} \leftarrow \emptyset$
$y(\ell) \leftarrow W \quad \forall \ell \in L$
$y(r) \leftarrow 0 \quad \forall r \in R$
shift weight and cheat
(a) too slow
(b) too much cheating
aug. paths are long
$[\tilde{O}(m/\epsilon)]$
repeat W times
APX Hopcroft-Karp
return $\mathcal{M}$
$O(mW/\epsilon)$
$\tilde{O}(m)$

Approximate scaling

1. Round up edge weights to powers of $(1+\epsilon)$:

$$\widetilde{w}(e) = (1+\epsilon)^{\lceil \log_{(1+\epsilon)} w(e) \rceil} \qquad e \in E$$

$$\widetilde{W} = \max_{e \in E} \widetilde{w}(e)$$

$(1-\epsilon)$-APX w/r/t $\widetilde{w} \Rightarrow (1-O(\epsilon))$-APX w/r/t w

Approximate scaling

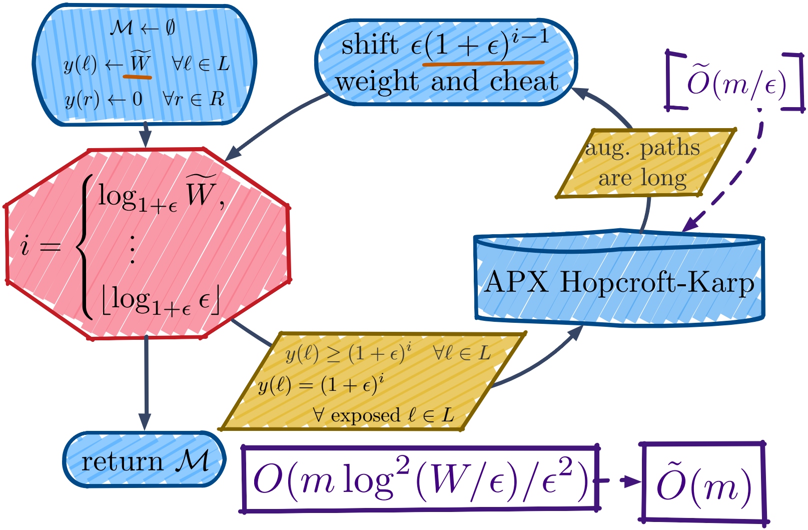
1. Round up edge weights to powers of $(1+\epsilon)$:

$$\widetilde{w}(e) = (1+\epsilon)^{\lceil \log_{(1+\epsilon)} w(e) \rceil} \qquad e \in E$$
$$\widetilde{W} = \max_{e \in E} \widetilde{w}(e)$$

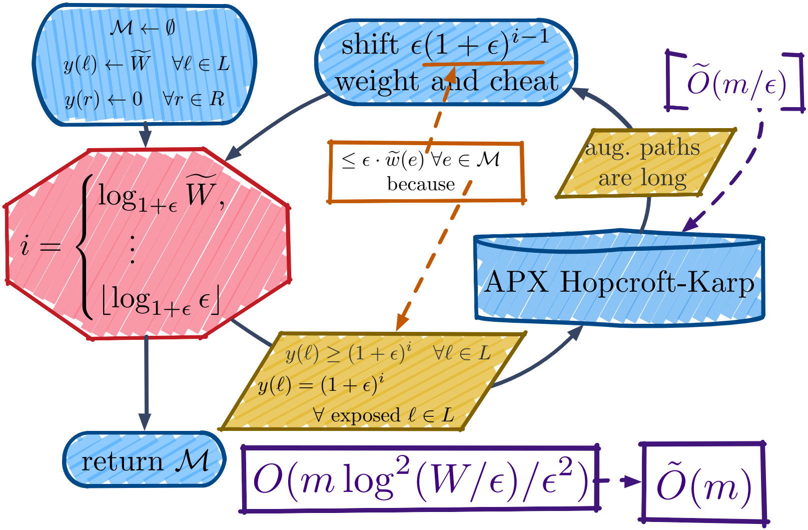
$(1-\epsilon)$-APX w/r/t $\widetilde{w} \Rightarrow (1 - O(\epsilon))$-APX w/r/t w

2. Operate in units of $\epsilon(1+\epsilon)^{i-1}$ for "scales"

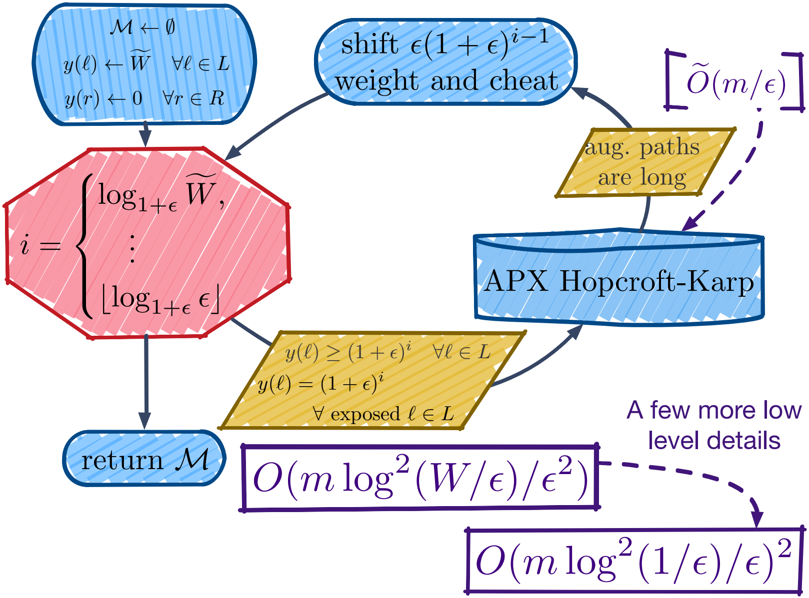
$$i = \log_{1+\epsilon} \widetilde{W}, \ldots, \lfloor \log_{1+\epsilon} \epsilon \rfloor$$



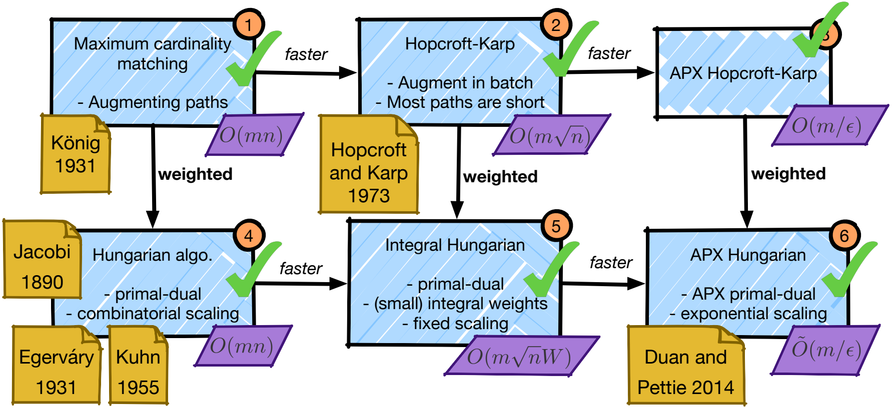
$\mathcal{M} \leftarrow \emptyset$
$y(\ell) \leftarrow \widetilde{W} \quad \forall \ell \in L$
$y(r) \leftarrow 0 \quad \forall r \in R$
shift $\epsilon(1+\epsilon)^{i-1}$ weight and cheat
$[\widetilde{O}(m/\epsilon)]$
$i = \begin{cases} \log_{1+\epsilon} \widetilde{W}, \\ \vdots \\ \lfloor \log_{1+\epsilon} \epsilon \rfloor \end{cases}$
aug. paths are long
APX Hopcroft-Karp
$y(\ell) \geq (1+\epsilon)^i \quad \forall \ell \in L$
$y(\ell) = (1+\epsilon)^i$
$\forall$ exposed $\ell \in L$
return $\mathcal{M}$
$O(m \log^2(W/\epsilon)/\epsilon^2)$
$\tilde{O}(m)$



$\mathcal{M} \leftarrow \emptyset$
$y(\ell) \leftarrow \widetilde{W} \quad \forall \ell \in L$
$y(r) \leftarrow 0 \quad \forall r \in R$
shift $\epsilon(1+\epsilon)^{i-1}$ weight and cheat
$\left[\widetilde{O}(m/\epsilon)\right]$
$\leq \epsilon \cdot \widetilde{w}(e) \, \forall e \in \mathcal{M}$ because
aug. paths are long
$i = \begin{cases} \log_{1+\epsilon} \widetilde{W}, \\ \vdots \\ \lfloor \log_{1+\epsilon} \epsilon \rfloor \end{cases}$
APX Hopcroft-Karp
$y(\ell) \geq (1+\epsilon)^i \quad \forall \ell \in L$
$y(\ell) = (1+\epsilon)^i$
$\forall$ exposed $\ell \in L$
return $\mathcal{M}$
$O(m \log^2(W/\epsilon)/\epsilon^2)$
$\tilde{O}(m)$



$\mathcal{M} \leftarrow \emptyset$
$y(\ell) \leftarrow \widetilde{W} \quad \forall \ell \in L$
$y(r) \leftarrow 0 \quad \forall r \in R$
shift $\epsilon(1+\epsilon)^{i-1}$ weight and cheat
$[\widetilde{O}(m/\epsilon)]$
$i = \begin{cases} \log_{1+\epsilon} \widetilde{W}, \\ \vdots \\ \lfloor \log_{1+\epsilon} \epsilon \rfloor \end{cases}$
aug. paths are long
APX Hopcroft-Karp
$y(\ell) \geq (1+\epsilon)^i \quad \forall \ell \in L$
$y(\ell) = (1+\epsilon)^i$
$\forall$ exposed $\ell \in L$
return $\mathcal{M}$
A few more low level details
$O(m \log^2(W/\epsilon)/\epsilon^2)$
$O(m \log^2(1/\epsilon)/\epsilon)^2$



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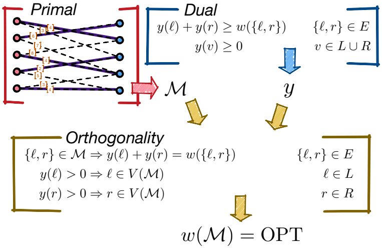
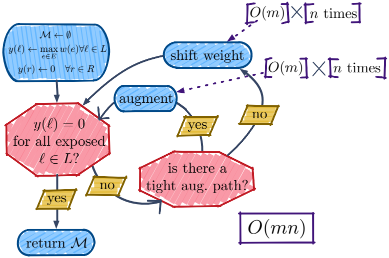
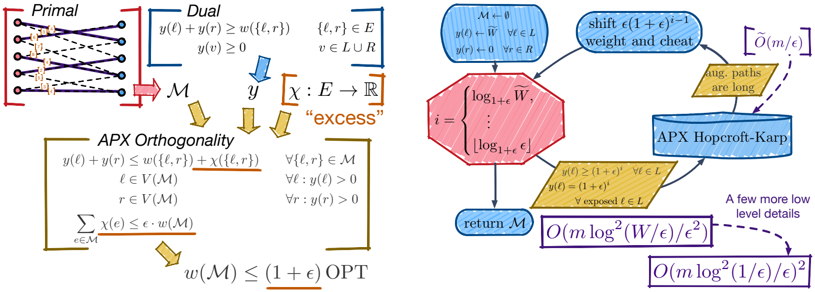
faster
APX Hungarian
- APX primal-dual
- exponential scaling
6
$\tilde{O}(m/\epsilon)$
Duan and Pettie 2014

exact algorithm





approximate algorithm



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