Metric TSP via the LP

Theory seminar, UIUC, March 5 2018

Joint work w/ Chandra Chekuri

Traveling Salesperson Problem (TSP)

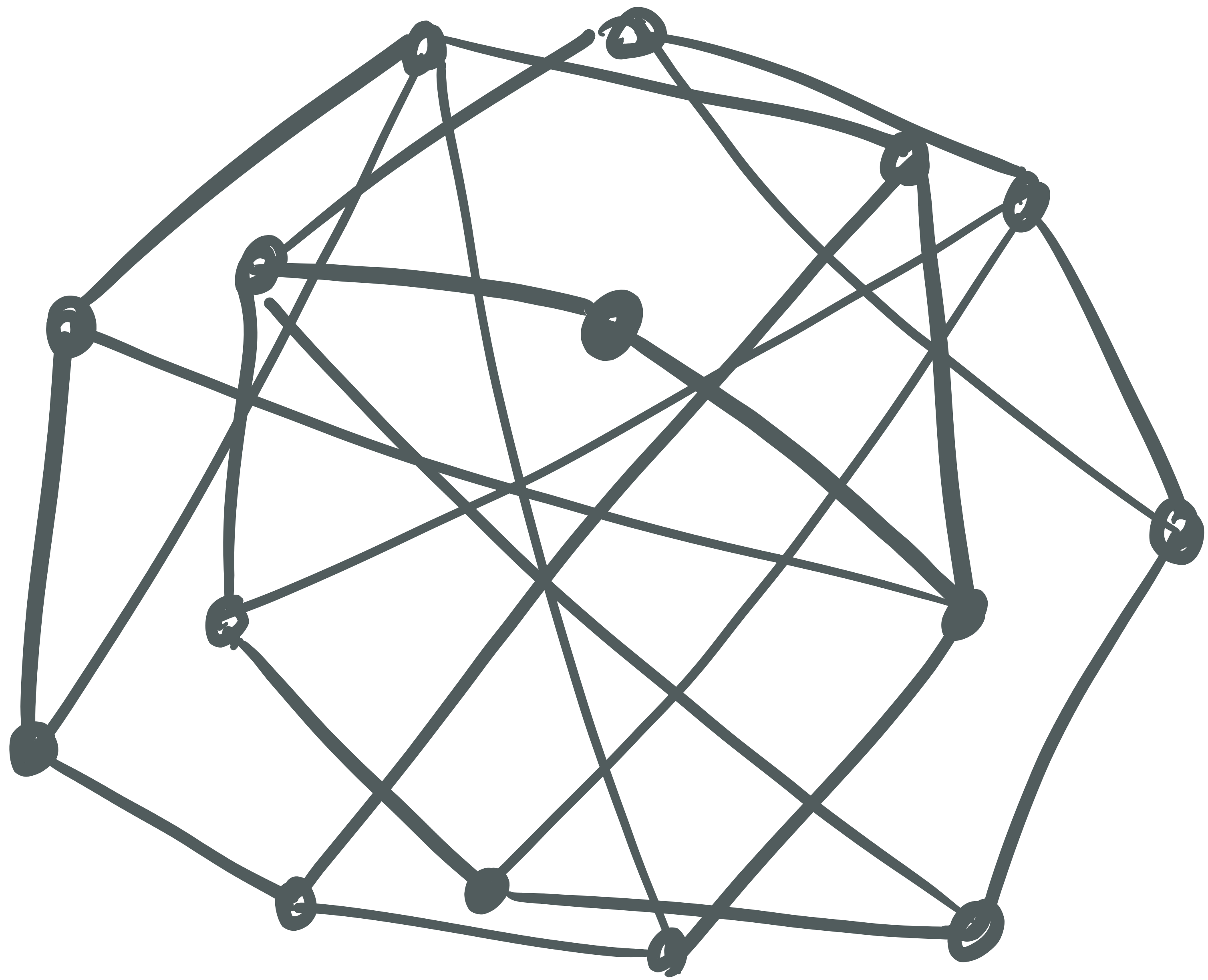
Input: Graph $G=(V,E)$, edge costs $c: E \to \mathbb{R}_{>0}$

(for this talk, all graphs are undirected)

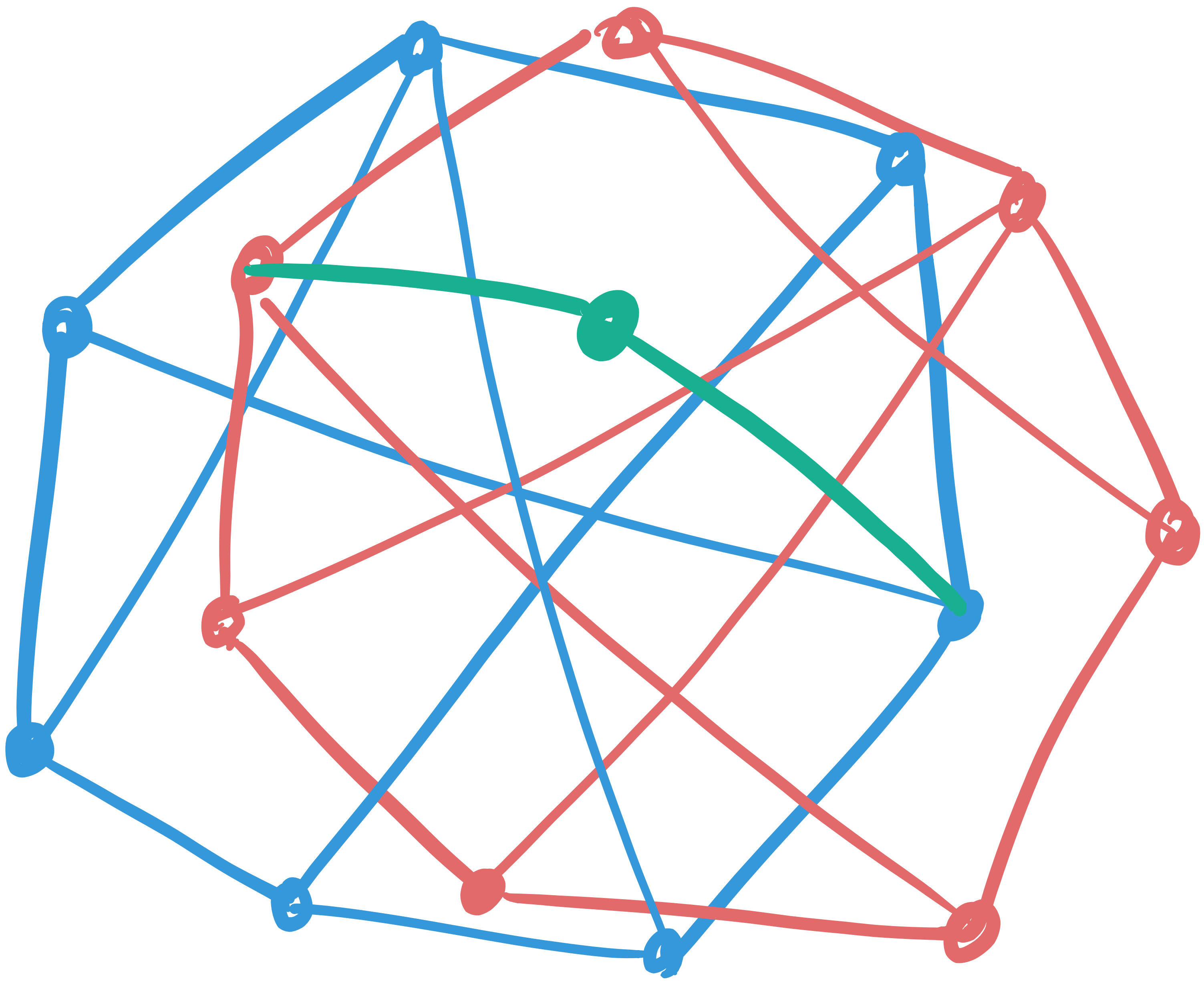
Goal: minimum cost Hamiltonian cycle

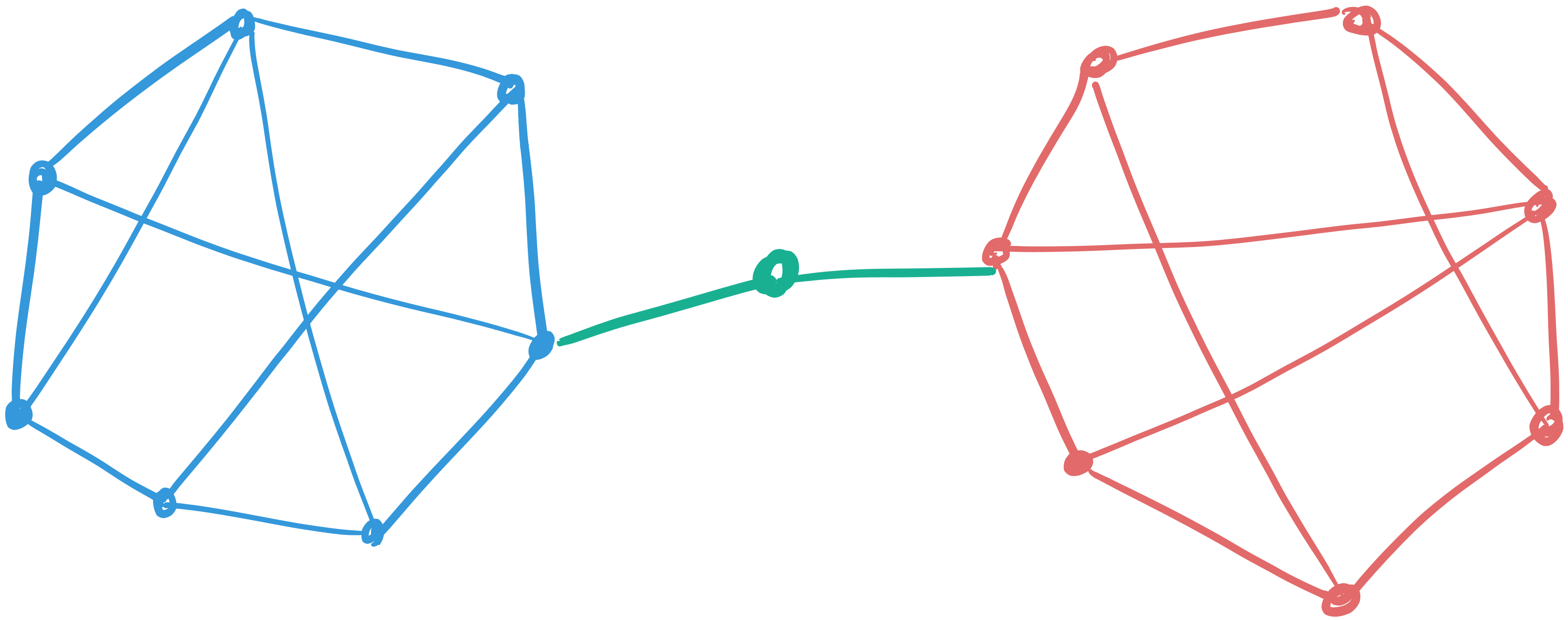
(visit every vertex exactly once)

Inapproximable: even deciding if there is a Hamiltonian cycle is NP-Hard (let alone the cost!)



Is there a Hamiltonian cycle?





(Nb)

Metric TSP

explicit version:

input: $G = K_n$, costs $c: E \to \mathbb{R}_{\geq 0}$ form a metric

goal: min-cost Hamiltonian cycle

implicit version:

input: $G = (V, E)$, costs $c: E \to \mathbb{R}_{\geq 0}$

goal: min-cost tour (visit each vertex at least once

Note: feasibility trivial

explicit = implicit

- make implicit problem explicit by computing APSP (all-pairs shortest paths)

explicit ≠ implicit

- explicit has input size $O(n^2)$
  implicit has input size $O(m)$
- computing APSP takes $\tilde{O}(mn)$ time to compute, $O(n^2)$ time to write down output

How do we approximate metric TSP?

Christofides' heuristic 1976

- $\frac{3}{2}$-APX
- simple (code coming up)
- state of the art

  (despite breakthroughs for special cases)

Christofides [1976]

1. Compute the MST $\mathcal{M}$ of G

2. Compute the minimum cost T-join $\mathcal{W}$ of G, where $T = \{\text{odd degree vertices in } \mathcal{M}\}$

3. The multiset $\mathcal{M} + \mathcal{W}$ is an Eulerian graph. Return an Eulerian tour of $\mathcal{M} + \mathcal{W}$

(T-joins defined next) (picture coming up)

T-joins

let $T \subseteq V$ w/ $|T|$ even

a **T-join** is a subgraph H of G for which T is the set of vertices of odd degree

min-cost T-join can be computed in poly time (details later)

Christofides [1976]

1. Compute the MST $\mathcal{M}$ of G
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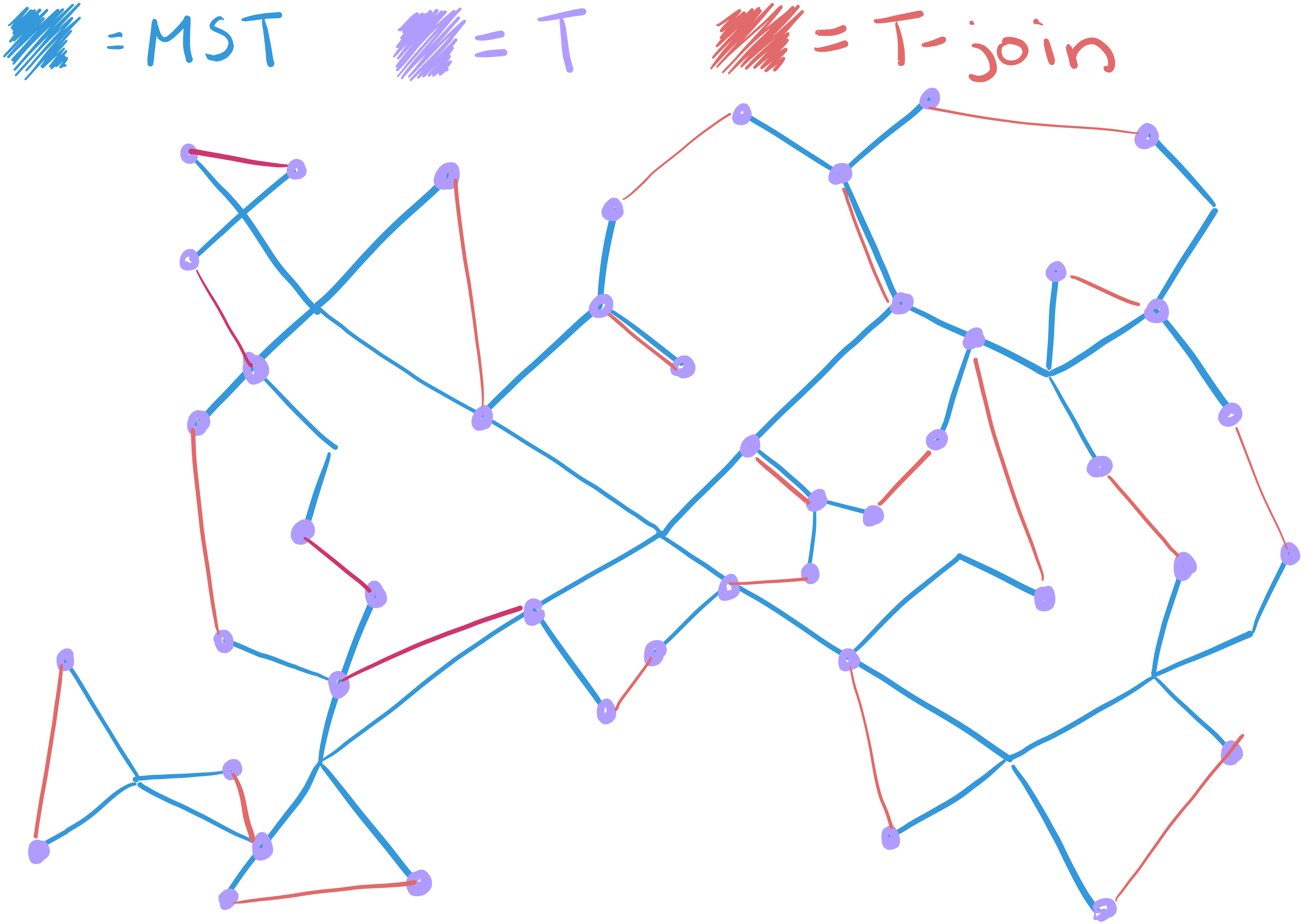
(picture coming up)



= MST
= T
= T-join

Proof:

Christofides returns MST M + T-join W

1. MST M contributes $(1-\frac{1}{n})OPT$

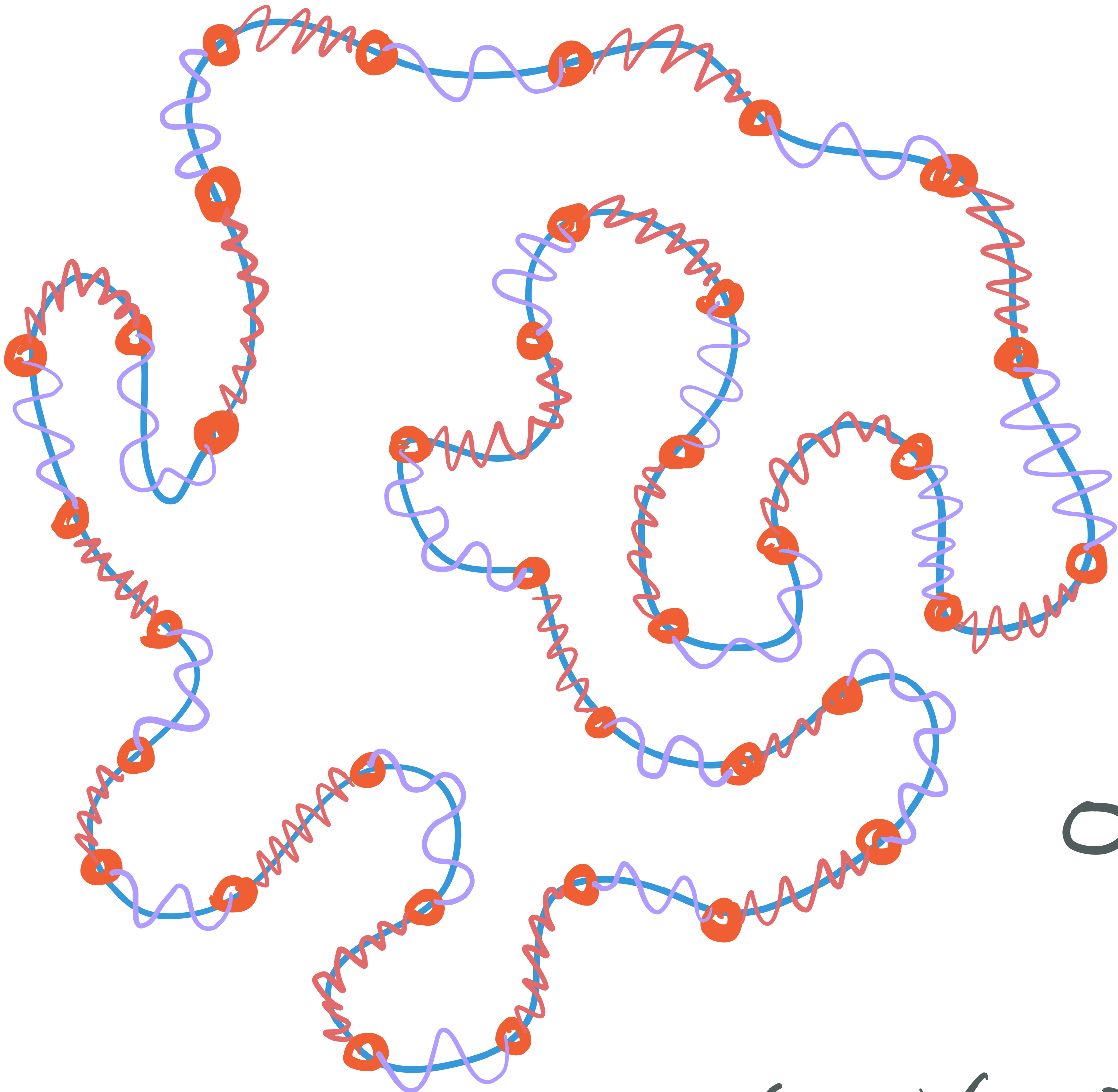
   (because M is min-cost connected subgraph, and TSP-(any edge) is connected)

2. T-join contributes $\frac{1}{2}OPT$

   (next slide)

$$(1-\tfrac{1}{n})OPT + \tfrac{1}{2}OPT \le \tfrac{3}{2}OPT \qquad \blacksquare$$

2. T-join contributes $\frac{1}{2}$OPT



= TSP

$\{\ \}$ = T

= 1st T-join

= 2nd T-join

$$\mathrm{OPT} = \mathrm{cost}(\ ) = \mathrm{cost}(\ ) + \mathrm{cost}(\ )$$

$$\Rightarrow \min\{\mathrm{cost}(\ ), \mathrm{cost}(\ )\} \le \tfrac{1}{2}\,\mathrm{OPT}$$

Christofides [1976]

Running time

1. Compute MST... $\tilde{O}(m)$

2. Compute minimum cost T-join... ??

3. Return Eulerian tour... $O(m)$

$\tilde{O}(m + (\text{time for T-join}))$

Perfect matchings

- $M \subseteq E$ is a perfect matching if each vertex is incident to exactly one edge in M.
- T-join for $T = V$
- Running times: (Gabow & Tarjan, 1991)

integer costs in $[-W, W]$ $\tilde{O}(m\sqrt{n}\log W)$

$(1+\varepsilon)$-APX for $\varepsilon > 0$ $\tilde{O}(m\sqrt{n}\log\frac{1}{\varepsilon})$

T-joins $\longrightarrow$ Perfect Matchings
(Edmonds, 1965)

1. Compute shortest paths for all pairs in T
2. Compute Perfect matching on T w/rt shortest paths

Running Time: $|T|$(Dijkstra) + (Perfect matching)

$= \hat{O}(m|T| + |T|^{2.5} \log W)$

or

$= \hat{O}(m|T| + |T|^{2.5} \log \frac{1}{\varepsilon})$

(Edmonds also handled negative weights)

Christofides [1976]

Running time

1. Compute MST... $\tilde{O}(m)$

2. Compute minimum cost T-join... $\tilde{O}(mn + n^{2.5})$

3. Return Eulerian tour... $O(m)$

$\tilde{O}(mn + n^{2.5})$

[$|T| = n$, $\varepsilon = \frac{1}{n}$,
use $\frac{1}{n}$-slack from MST]

T-joins → Perfect matchings #2

(Berman, Kahng, Vidhani, Zelikovsky 1999)

Theorem: In $O(m)$ time, the min-cost T-join can be reduced to min-cost perfect matching in a graph w/ $O(m)$ nodes and $O(m)$ edges

$\Rightarrow \tilde{O}(m^{1.5} \log W)$, $\tilde{O}(m^{1.5} \log \frac{1}{\varepsilon})$

T-joins to (sparse) perfect matchings

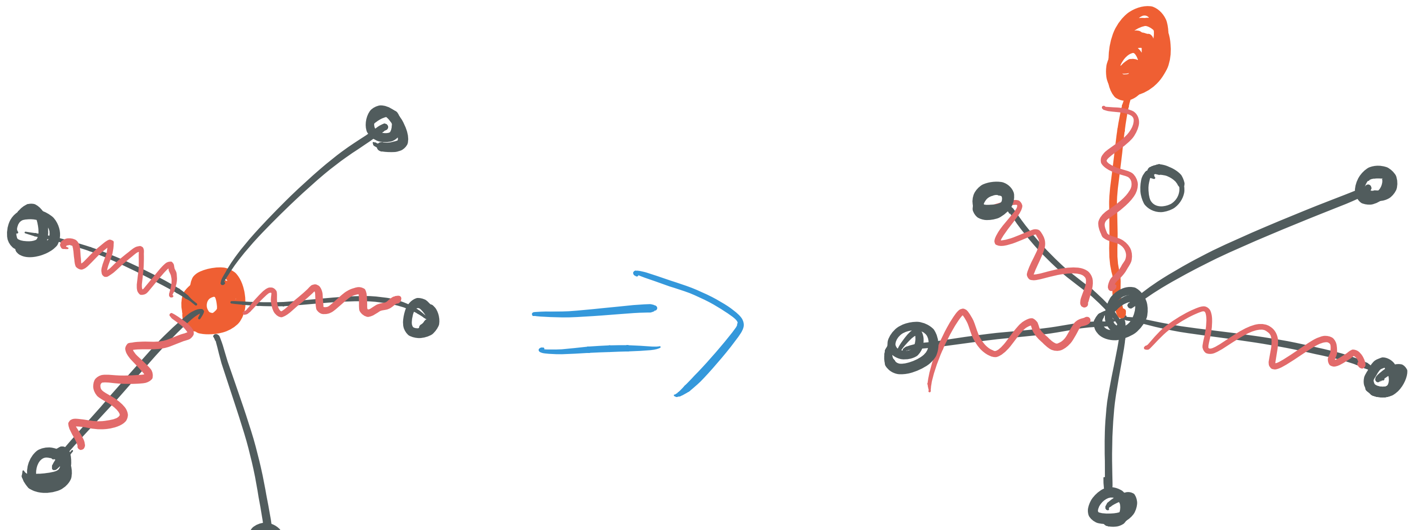
3 steps (pictures to follow)

1. all $t \in T$ have degree 1

2. all vertices have degree ≤ 3

3. reduction to perfect matching

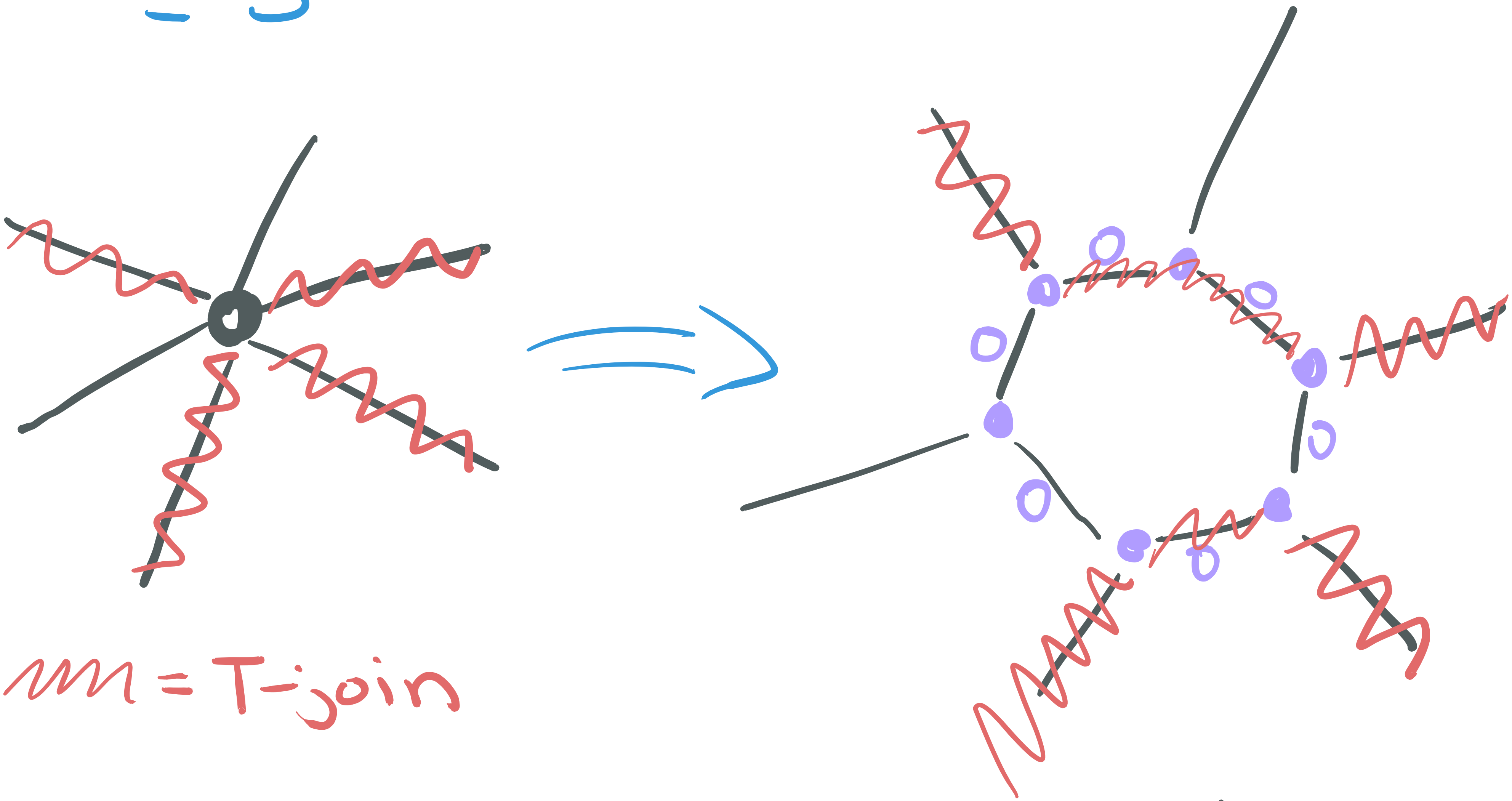
1. make every $t \in T$ have degree 1



= T-join

= vertex in T

2. make all vertices have degree ≤ 3

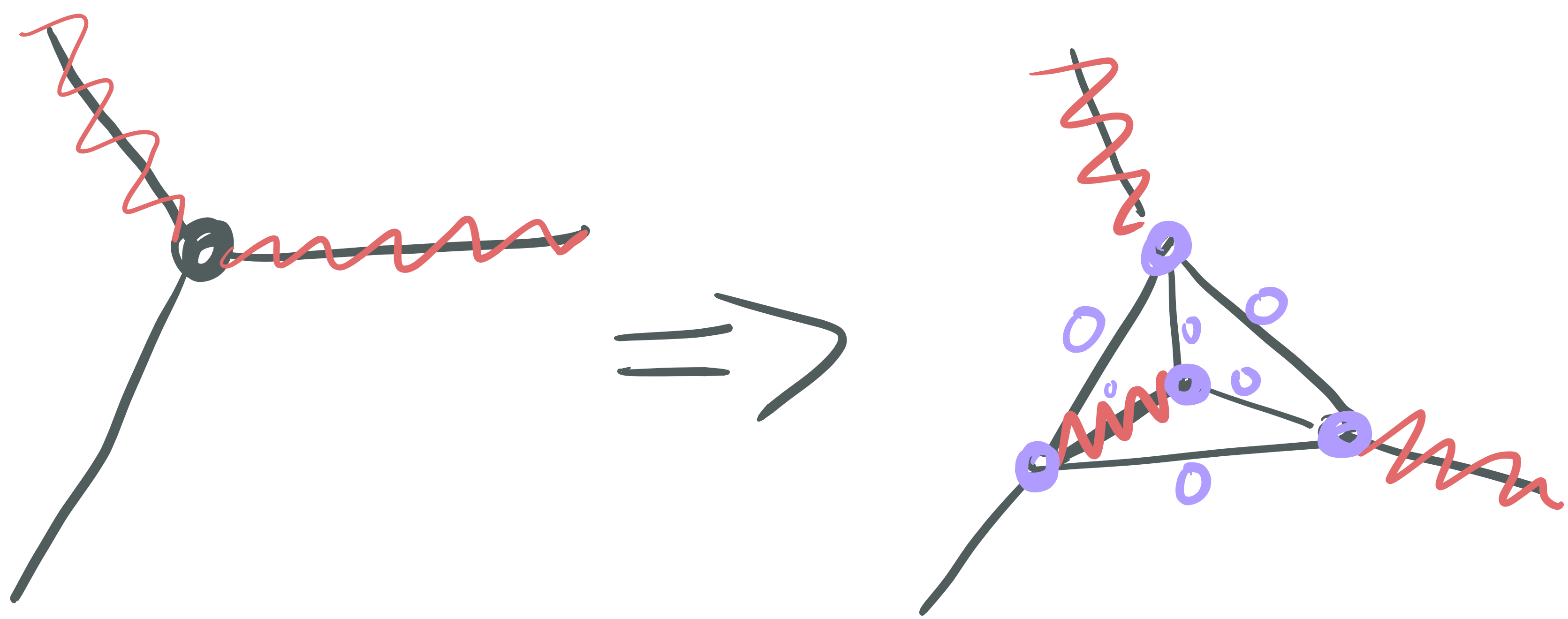


= T-join

= new vertices

$\Rightarrow O(m)$ edges, $O(m)$ vertices

Reduction to perfect matching



$\Rightarrow$ prefect matching w/ $O(m)$ edges,
$O(m)$ vertices

Christofides [1976]

Running time

1. Compute MST... $\tilde{O}(m)$

2. Compute minimum cost T-join... $\tilde{O}(m^{1.5})$

3. Return Eulerian tour... $O(m)$

$\tilde{O}(m^{1.5})$

[$\varepsilon = \frac{1}{n}$,
use $\frac{1}{n}$-slack from MST]

Running times for metric TSP

Metric	$3/2$-APX
Shortest paths	$\tilde{O}(mn + n^{2.5})$
	$\tilde{O}(m^{1.5})$
Explicit	$\tilde{O}(n^{2.5})$

Our result

let $\varepsilon > 0$:

$(1+\varepsilon)\frac{3}{2}$-APX to metric TSP in

$\tilde{O}\left(\frac{m}{\varepsilon^2} + \frac{n^{1.5}}{\varepsilon^3}\right)$ randomize time

w/h/p

Running times for metric TSP

Metric	$3/2$-APX	$(1+\varepsilon)\frac{3}{2}$-APX
Shortest paths	$\tilde{O}(mn + n^{2.5})$ $\tilde{O}(m^{1.5})$	$\boxed{\tilde{O}\left(\frac{m}{\varepsilon^2} + \frac{n^{1.5}}{\varepsilon^3}\right)}$
Explicit	$\tilde{O}(n^{2.5})$	$\boxed{\tilde{O}\left(\frac{n^2}{\varepsilon^2} + \frac{n^{1.5}}{\varepsilon^3}\right)}$

APX-Christofides (sketch)

1. Compute MST M
2. Sparsify graph (more details to come)
3. compute min-cost T-join J in sparsified graph, for $T = \{$odd degree vertices in $M\}$
4. Return Eulerian Tour of $M + J$

2. Sparsify graph

(a) solve LP relaxation 2ECSS

(next slide)

(b) use solution as edge weights and sparsify

(via Benczur-Karger, 2015)

SE (subtour elimination)

minimize $\sum_{e \in E} c_e y_e$ over $y \in \mathbb{R}^E$

s.t $\sum_{e \in \delta(v)} y_e = 2$ for all $v \in V$

degree 2 for all vertices

$\sum_{e \in \delta(U)} y_e \geq 2$ for all $\emptyset \subsetneq U \subsetneq V$

$y_e \in [0,1]$ for all $e \in E$

eliminates subtours

SE (subtour elimination)

$$\text{minimize } \sum_{e\in E} c_e y_e \text{ over } y \in \mathbb{R}^E$$

$$\text{s.t } \sum_{e\in\delta(v)} y_e = 2 \text{ for all } v\in V$$

$$\sum_{e\in\delta(U)} y_e \geq 2 \text{ for all } \emptyset \subsetneq U \subsetneq V$$

$$y_e \in [0,1] \text{ for all } e\in E$$

[Wolsey 1980] if y is feasible to $\,$, then $(1-\frac{1}{n})$ lies in the spanning tree polytope.

2ECSS (2-edge connected spanning subgraph)

$$\text{minimize} \sum_{e \in E} c_e y_e \text{ over } y \in \mathbb{R}^E$$

$$\text{s.t.} \sum_{e \in \delta(U)} y_e \geq 2 \quad \forall \emptyset \subsetneq U \subsetneq V$$

$$y_e \geq 0 \quad \forall e \in E$$

eliminates subtours

2ECSS (2-edge connected spanning subgraph)

$$\text{minimize} \sum_{e\in E}' c_e y_e \quad \text{over } y \in \mathbb{R}^E$$

$$\text{s.t.} \sum_{e\in\delta(U)}' y_e \geq 2 \quad \forall \emptyset \subsetneq U \subsetneq V$$

$$y_e \geq 0 \quad \forall e \in E$$

eliminates subtours

equivalent to subtour elimination LP.

(if a vertex has degree >2, shortcut it)

$\uparrow$ Cunningham; Bertsimas and Goemans

2ECSS (2-edge connected spanning subgraph)

minimize $\sum_{e\in E} c_e y_e$ over $y \in \mathbb{R}^E$

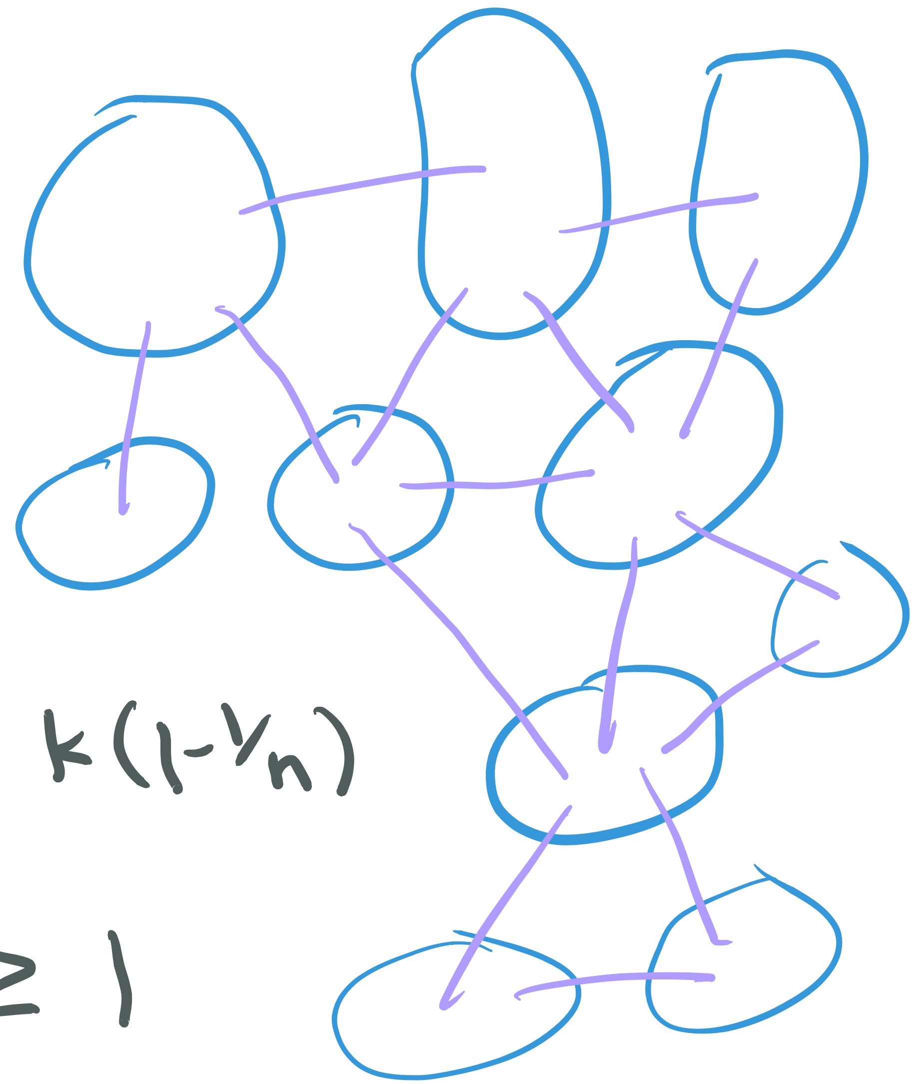
s.t. $\sum_{e\in\delta(U)} y_e \geq 2 \quad \forall \emptyset \subsetneq U \subsetneq V$

$y_e \geq 0 \quad \forall e \in E$

eliminates subtours

If y is a feasible point to $\hat{\int}$, then $(1-\frac{1}{n})$ lies in the dominant of the spanning tree polytope.

- consider any k-cut
- each part contributes $\geq 2 \cdot (1 - 1/n)$ edges (w/r/t y)



$$\# \text{edges} \geq \frac{2 \cdot k \cdot (1 - 1/n)}{2} = k(1 - 1/n)$$

$$\Rightarrow \frac{\# \text{edges}}{k-1} \geq \frac{k(1-1/n)}{k-1} \geq 1$$

$\Rightarrow$ by Tutte-Nash-Williams we can pack 1 fractional spanning tree

$$\text{minimize } \sum_{e\in E} c_e y_e \text{ over } y \in \mathbb{R}^E$$

$$\text{s.t. } \sum_{e\in\delta(U)} y_e \geq 2 \quad \forall \emptyset \subsetneq U \subsetneq V$$

eliminates subtours

$$y_e \geq 0 \quad \forall e \in E$$

Theorem (Chekuri-Q '17)

In $\tilde{O}(m/\varepsilon^2)$ randomized time, one can compute a $(1+\varepsilon)$-APX to 2ECSS w/ high probability.

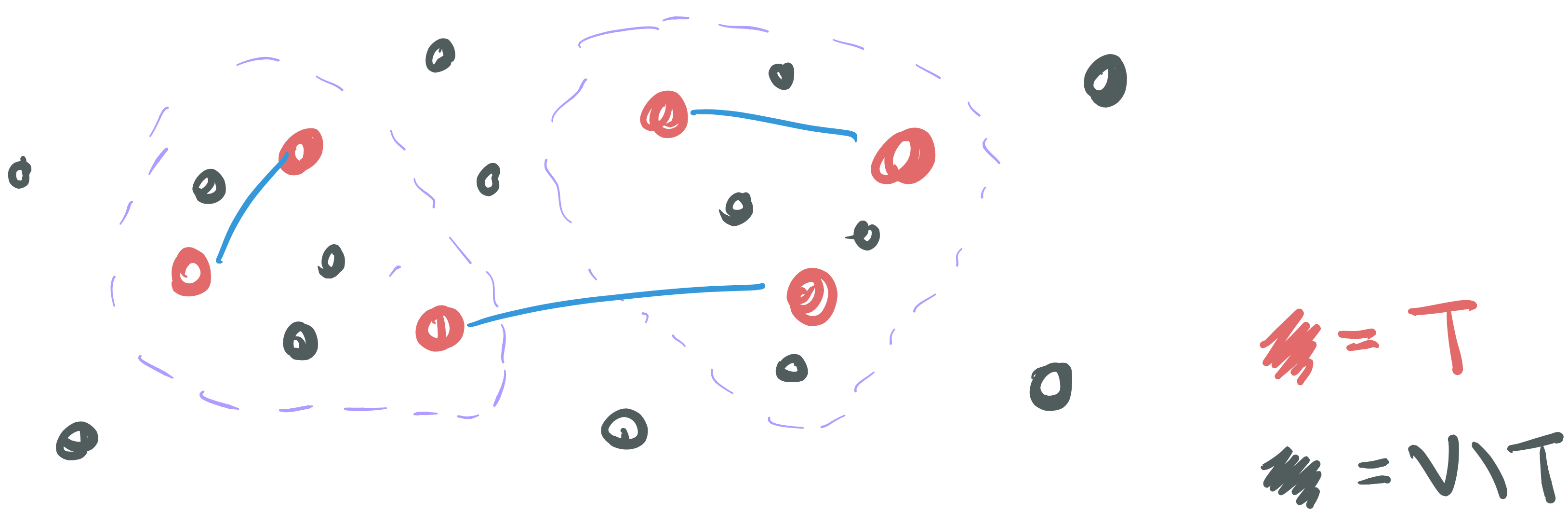
The dominant of the T-join polytope

(Edmonds & Johnson, 1973)

$x \in \mathbb{R}^E$ such that

$$\sum_{e \in \delta(S)} x_e \geq 1 \quad \forall\ S \subseteq V,\quad |S \cap T| \text{ odd}$$

$$x_e \geq 0 \quad \forall\ e \in E$$



2ECSS

$$\sum_{e \in \delta(U)} y_e \geq 2 \quad \forall \phi \subsetneq U \subsetneq V$$

$$y_e \geq 0 \quad \forall e \in E$$

Dominant of T-join

$$\sum_{e \in \delta(S)} x_e \geq 1 \quad \forall S \subseteq V, |S \cap T| \text{ odd}$$

$$x_e \geq 0 \quad \forall e \in E$$

[Wolsey 1980]. if $y \in \mathbb{R}^n$ is feasible for 2ECSS, then $y/2$ is in dominant of the T-join polytope for any even $T \subseteq V$

$\Rightarrow$ we can compute T-join in support of y!!

Sparsifying 2ECSS

$$\text{minimize } \sum_{e\in E}' c_e y_e \text{ over } y \in \mathbb{R}^E$$

$$\text{s.t. } \sum_{e\in\delta(U)}' y_e \geq 2 \quad \forall \phi \subsetneq U \subsetneq V$$

$$y_e \geq 0 \quad \forall e \in E$$

goal: given $y \in 2ECSS$, compute $x \in 2ECSS$ with similar cost and $|\text{support}(x)|$ small.

Sparsifying 2ECSS

minimize $\sum_{e \in E} c_e y_e$ over $y \in \mathbb{R}^E$

s.t. $\sum_{e \in \delta(U)} y_e \geq 2 \quad \forall \emptyset \subsetneq U \subsetneq V$

$y_e \geq 0 \quad \forall e \in E$

Interpret y as edge weights. Then x really needs to preserve the size of each cut

$\Rightarrow$ "cut sparsifiers"

goal: given $y \in$ 2ECSS, compute $x \in$ 2ECSS with similar cost and $|\text{support}(x)|$ small.

Cut sparsify

let $\varepsilon > 0$

given a weighted graph $G = (V, E, y)$, a cut sparsifier is a (sub-)graph $H = (V, E', x)$ that preserves each cut to $(1 \pm \varepsilon)$ factor.

(eg Charlie's talk)

Benczur-Karger algo finds H w/ $|E'| = O\left(\frac{n \log n}{\varepsilon^2}\right)$ in $\tilde{O}(m/\varepsilon^2)$ randomized time.

Sparsifying 2ECSS

$$\text{minimize } \sum_{e \in E} c_e y_e \text{ over } y \in \mathbb{R}^E$$

$$\text{s.t. } \sum_{e \in \delta(U)} y_e \geq 2 \quad \forall \emptyset \subsetneq U \subsetneq V$$

$$y_e \geq 0 \quad \forall e \in E$$

<u>Theorem</u> one can compute $(1+\varepsilon)$-APX y to 2ECSS w/ $|\text{support } y| = O\left(\frac{n \log n}{\varepsilon^2}\right)$ in $\tilde{O}(m/\varepsilon^2)$ randomized time w/h/p.

(minor adjustment to preserve cost)

Computing APX T-join

1. compute $(1+\varepsilon)$-APX x to 2ECSS
2. Apply Benczur-Karger to get $(1+\varepsilon)$-APX y to 2ECSS w/ $\|y\|_0 = O\left(\frac{n\log n}{\varepsilon^2}\right)$
3. y is in dominant of T-Join. compute min-cost T-Join in $H = (V, \text{support}(y), y)$

$\Rightarrow$ T-join J w/ cost $\sum_{e \in J} c_e \leq \frac{(1+\varepsilon)}{2}(\text{2ECSS})$

Computing APX T-join	Running time
1. compute 2ECSS...	$\tilde{O}(m/\varepsilon^2)$
2. Apply Benczur-Karger...	$\tilde{O}(m/\varepsilon^2)$
3. compute min cost T-Join in $H = (V, \text{support}(y), y)$	$\tilde{O}(n^{1.5}/\varepsilon^2)$

use sparse reduction of Berman et al w/ $O(m^{3/2})$ running time $[m = \tilde{O}(n/\varepsilon^2)]$

APX-Christofides

1. Compute MST M

2. Sparsify graph (more details to come)

$\tilde{O}\left(\frac{m}{\varepsilon^2} + \frac{n^{3/2}}{\varepsilon^2}\right)$ time

3. compute min-cost T-join J in sparsified graph, for $T = \{$odd degree vertices in $M\}$

4. Return Eulerian Tour of $M + J$

Theorem

In $\hat{O}\left(\frac{m}{\varepsilon^2} + \frac{n^{3/2}}{\varepsilon^3}\right)$ randomized time, one can compute $(1+\varepsilon)\frac{3}{2}$* APX to metric TSP w/h/p.

* cost is $\leq (1+\varepsilon)\frac{3}{2}(2ECSS)$

Running times for metric TSP

Metric	$3/2$-APX	$(1+\varepsilon)\frac{3}{2}$-APX
Shortest paths	$\tilde{O}(mn+n^{2.5})$ $\tilde{O}(m^{1.5})$	$\tilde{O}\left(\frac{m}{\varepsilon^2}+\frac{n^{1.5}}{\varepsilon^3}\right)$
Explicit	$\tilde{O}(n^{2.5})$	$\tilde{O}\left(\frac{n^2}{\varepsilon^2}+\frac{n^{1.5}}{\varepsilon^3}\right)$

Extensions

- Also accelerates best-of-many Christofides heuristic

- similar results for path TSP (forthcoming)

THANKS!